BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

CLASSE TROISIÈME

VOLUME II
NUMÉRO 7

VARSOVIE 1954

PRINTED IN POLAND

PAŃSTWOWE WYDAWNICTWO NAUKOWE — WARSZAWA 1954

Nakład 1019 + 100 egz.	*Rękopis dostarczono 8. VI. 1954*
Ark. wyd. 3·7, druk. 3·5	*Podpisano do druku 9. VIII. 1954*
Papier bezdrzewny sat. 80 g kl. III	*Druk ukończono 17. VIII. 1954*
Format B5, 70×100 cm	*Zam. prod. 342/54 Cena zł 5,—*

KRAKOWSKA DRUKARNIA NAUKOWA, KRAKÓW, UL. CZAPSKICH 4

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. II, No. 7, 1954

MATHÉMATIQUE

Sur les espaces riemanniens dégénérés

par

C. JANKIEWICZ

Présenté par W. RUBINOWICZ à la séance du 22 Février 1954

En juin 1953, au cours d'une conférence de physique théorique à Wrocław R. S. Ingarden a proposé d'appliquer, à la théorie du champ, les espaces riemanniens dégénérés pour lesquels le déterminant de la forme métrique

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{1}$$

est identiquement nul. On appellera défaut de l'espace la grandeur $d = n - m$, où n représente le nombre des dimensions de l'espace et m le rang de la matrice $g_{\alpha\beta}$.

Le but de ce travail est de construire une classe de connexions affines liée aussi naturellement que possible à la forme quadratique (1) pour n et m quelconques. Dans le cas $d = 1$, ce problème a été étudié par A. P. Norden [1], [2]. Cependant les résultats qu'il a acquis ne peuvent être généralisés directement pour $d > 1$.

Détermination des coefficients de connexion par $g_{\alpha\beta}$ et $\underset{E}{g}^{\alpha}$

Supposons qu'il existe, dans l'espace à n dimensions L_n à connexion affine (asymétrique) (voir p. ex. [2], § 32), un tenseur métrique $g_{\alpha\beta}(x^\gamma)$ *) dont le rang soit en chaque point de l'espace $m < n$ (on considére le domaine de l'espace, dans lequel le rang de la matrice $g_{\alpha\beta}$ est constant). Il en résulte que le système d'équations

$$g_{\alpha\beta}\, g^\beta = 0 \tag{2}$$

a $d = n - m$ solutions $\underset{E}{g}^{\beta}$ **), linéairement indépendantes, qui déterminent dans l'espace tangent, en chaque point, un sous-espace vectoriel à d dimensions. Choisissons une numération des coordonnées de manière que $\det(g_{ab}) \neq 0$.

*) Les indices en minuscules grecques parcourent les valeurs $1, \ldots, n$.

**) Les indices en minuscules latines parcourent les valeurs $1, \ldots, m$, et ceux en majuscules latines les valeurs $m+1, \ldots, n$.

Il est facile de prouver qu'on aura alors, pour les vecteurs linéairement indépendants $\underset{E}{g}^{\beta}$, $\det(\underset{E}{g}^{A}) \neq 0$.

On peut écrire la solution du système (2) sous la forme

$$\underset{E}{g}^{b} = -g_{aA}\, g^{ab}\, \underset{E}{g}^{A}. \tag{3}$$

En fixant arbitrairement d^2 fonctions $\underset{E}{g}^{A}$ (tout en tenant compte de la condition $\det(\underset{E}{g}^{A}) \neq 0$) on obtiendra toujours d solutions linéairement indépendantes $\underset{E}{g}^{\beta}$.

Supposons ensuite que le tenseur $g_{\alpha\beta}$ et les vecteurs $\underset{E}{g}^{\beta}$ se déplacent parallèlement par rapport à la connexion recherchée

$$g_{\alpha\beta;\gamma} = 0\,{}^{*)}, \tag{4 a}$$

$$\underset{E}{g}^{\beta}{}_{;\gamma} = 0. \tag{4 b}$$

L'hypothèse (4 a) signifie qu'un espace est métrique dans le sens employé par Schouten et par Struik [3]; (4 b) est une généralisation naturelle de cette hypothèse pour les espaces dégénérés. On peut remplacer les hypothèses (4) par des hypothèses plus générales qui conduisent aux espaces "semi-métriques" (de Weyl) — comme l'a fait Norden pour $d=1$ [1], [2] — et aux espaces "généraux" [3]. Néanmoins on se bornera aux espaces métriques, car ces généralisations ne paraissent pas intéressantes pour les applications mentionnées plus haut.

Pour les coefficients de connexion, les relations suivantes résultent des hypothèses (4):

$$\Gamma^{r}_{ab} = \left\{ {r \atop ab} \right\} + g^{er}(S^{s}_{ab}\, g_{se} - S^{s}_{ae}\, g_{sb} - S^{s}_{be}\, g_{sa}) + g^{er}(S^{A}_{ea}\, g_{Ab} + S^{A}_{eb}\, g_{Aa} - \Gamma^{A}_{(ab)}\, g_{Ae})$$

$$\begin{aligned}
\Gamma^{a}_{AB} &= \overset{E}{g}_{B}\, \underset{E}{g}^{a}{}_{,A} + g^{s}_{B}\, \overset{E}{g}_{A}\, \underset{E}{g}^{a}{}_{,s} + \Gamma^{a}_{se}\, g^{s}_{B}\, g^{e}_{A} \\
\Gamma^{A}_{BD} &= -\overset{E}{g}_{D}\, \underset{E}{g}^{A}{}_{,B} + g^{s}_{D}\, \overset{E}{g}_{B}\, \underset{E}{g}^{A}{}_{,s} + \Gamma^{A}_{se}\, g^{s}_{B}\, g^{e}_{D} \\
\Gamma^{a}_{bA} &= -\overset{E}{g}_{A}\, \underset{E}{g}^{a}{}_{,b} - \Gamma^{a}_{bs}\, g^{s}_{A}, \qquad \Gamma^{A}_{bB} = -\overset{E}{g}_{B}\, \underset{E}{g}^{A}{}_{,b} - \Gamma^{A}_{bs}\, g^{s}_{B}
\end{aligned} \tag{5}$$

où

$$\left\{ {r \atop ab} \right\} = \frac{1}{2} g^{rs}(g_{bs,a} + g_{sa,b} - g_{ab,s}), \qquad S^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{[\beta\gamma]} = \frac{1}{2}(\Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta})$$

$$\Gamma^{\alpha}_{(\beta\gamma)} = \frac{1}{2}(\Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\gamma\beta}), \qquad \underset{E}{g}^{A}\, \overset{E}{g}_{B} = \delta^{A}_{B}, \qquad g^{ar}\, g_{rb} = \delta^{a}_{b}, \qquad \underset{A}{g}^{a}\, \overset{A}{g}_{B} = g^{a}_{B}\,{}^{**)}.$$

*) Le point-virgule devant l'indice signifie la différentiation covariante suivant la coordonnée avec cet indice. Analogiquement la virgule signifierait la différentiation ordinaire.

**) $\delta^{\alpha}_{\beta} = 1$ pour $\alpha = \beta$ et 0 pour $\alpha \neq \beta$.

Les relations (5) montrent qu'on peut fixer arbitrairement:

1) $\frac{1}{2}m^2(m-1)$ fonctions S^a_{be},
2) $m^2(n-m)$ fonctions Γ^A_{ab},
3) $(n-m)^2$ fonctions $\underset{E}{g^A}$.

Le choix de ces fonctions est indépendant du choix du tenseur métrique. Toutefois, pour déterminer un tenseur métrique aux propriétés exigées, il faut donner $nm-\frac{1}{2}(m^2-m)$ fonctions indépendantes. Donc pour une détermination univoque de la connexion, il faut $\frac{1}{2}m(m-1)(2n-m-1)+n^2$ fonctions.

Les espaces réductibles

Les formules (5) ont lieu dans chaque systéme de coordonnées où la condition $\det(g_{ab})\neq 0$ est satisfaite. En particulier cette condition subsistera après la transformation réductible

(6) $$x^{a'}=x^{a'}(x^a),\ x^{A'}=x^{A'}(x^A).$$

Par rapport à la transformation (6) toutes les expressions qui se trouvent dans les formules (5) sont soit des tenseurs, soit des coefficients de connexion respectivement dans un sous-espace x^a à m dimensions, dans un sous-espace x^A à d dimensions, ou bien dans les deux sous-espaces simultanément (comme objects géométriques intermédiaires, voir [2], § 37) suivant le genre de leurs indices. Γ^a_{be} et Γ'^A_{BE} sont coefficients de connexion; toutes les autres expressions sont des tenseurs.

Si dans un espace métrique dégénéré il existe un système de coordonnées, où

(7) $$(g_{\alpha\beta})=\begin{pmatrix} g_{ab} & 0\\ 0 & 0\end{pmatrix},$$

on apellera un tel espace réductible. Nous allons montrer que, pour qu'un espace soit réductible, il faut et il suffit que

(8) $$g^a_{[A}\,g^s_{B];a}+g^s_{[A;B]}=S^E_{AB}\,g^s_E$$

(il est évident que cette condition est invariante par rapport aux transformations réductibles).

Considérons le système de formes de Pfaff

(9) $$g_{s\alpha}dx^\alpha=0.$$

Pour qu'il existe un système de coordonnées où a lieu (7), il faut et il suffit que le système de formes (9) soit complètement intégrable [4]. En transformant (9) en la forme (profitant de (3))

(10) $$dx^s-g^s_A\,dx^A=0$$

et en appliquant le théorème de Frobenius [4] sur l'intégrabilité complète du système (10) on a la condition

$$g^a_A g^s_{B,a} - g^a_B g^s_{A,a} = g^s_{A,B} - g^s_{B,A}. \tag{11}$$

En exprimant les dérivées ordinaires par les dérivées covariantes et par les coefficients de connexion on obtient la condition exigée (8).

On trouvera le système de coordonnées où a lieu (7), en prenant, comme nouvelles coordonnées $x^{a'}$, m intégrales indépendantes du système (9), [4]. On a pour ces coordonnées $g_{ab,B} = 0$ et $\underset{A}{g^a} = 0$.

Parmi les espaces réductibles on peut distinguer ceux qui satisfont aux conditions supplémentaires

$$\underset{E}{g^A_{,a}} = 0,\ S^s_{ab} = 0,\ \Gamma^A_{ab} = 0.$$

Ces espaces se décomposent en deux sous-espaces indépendants l'un de l'autre, à m et d dimensions respectivement, pour lesquels

$$\Gamma^r_{ab} = \left\{ {r \atop a\,b} \right\},\ \Gamma^A_{BD} = -\overset{E}{g_D}\underset{E}{g^A_{,B}}, \quad g_{ab,A} = 0,\ \underset{E}{g^A_{,a}} = 0.$$

En calculant pour ces espaces le tenseur de courbure de Riemann-Christoffel $R^\sigma_{\alpha\beta\gamma}$ et le tenseur de torsion $S^\alpha_{\beta\gamma}$, on verra que les parties intermédiaires de ces tenseurs sont nulles; différents de zéro peuvent être exclusivement

$$R^s_{abe},\ S^A_{BD},\ \text{et}\ R^D_{ABE}.$$

Pour $d=1$, tout espace est réductible. En introduisant alors le système canonique [1] et en posant $S_{\alpha\beta} = 0$ et $\omega_\alpha = 0$, on obtient les formules sur les coefficients de connexion, calculées par Norden [1], [2].

Je veux adresser ici mes remerciements aux professeurs W. Ślebodziński et R. S. Ingarden pour l'aide qu'ils m'ont accordée dans l'élaboration de ce travail.

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OUVRAGES CITÉS

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[3] Schouten J. A. et Struik D. J., *Einführung in die neueren Methoden der Differentialgeometrie,* I, Groningen 1938, § 8.

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POLONAISE DES SCIENCES
Cl. III — Vol. II, No. 7, 1954

MATHEMATIK

Über die Einbettung eines Finslerschen Raumes in einem Minkowskischen Raum

von

R. S. INGARDEN

Vorgelegt von W. RUBINOWICZ in der Sitzung am 22. Februar 1954

Im Jahre 1947 hat der Verfasser in einer Abhandlung über die stigmatischen Abbildungen im Elektronenmikroskop [1] das auch separat veröffentlichte Problem [2] gestellt, ob für einen jeden Finslerschen n-dimensionalen Raum [3] [4] ein Minkowskischer Raum*) existiert, in welchem sich der erstere einbetten liesse. Der Begriff der Einbettung wurde hierbei analog zum Begriff der Einbettung eines Riemannschen Raumes in einem euklidischen Raum gefasst und lässt sich auf folgende Weise definieren.

Der Finslersche Raum soll mit der für die Entfernung beliebiger unendlich naher Punkte x_μ und $x_\mu + dx_\mu$**) geltenden Formel

$$ds = F(x_\mu, dx_\mu) \tag{1}$$

„im Kleinen" gegeben sein, wobei die Funktion $F(x_\mu, x'_\mu)$ im betrachteten Variabilitätsbereich ihrer Argumente analytisch und regulär ist und in diesem Bereiche die beiden Bedingungen

$$F(x_\mu, kx'_\mu) = kF(x_\mu, x'_\mu), \qquad k > 0, \tag{2}$$

und

$$F_1 = \frac{-1}{(x'_\mu x'_\mu)^2} \begin{vmatrix} \dfrac{\partial^2 F}{\partial x'_\mu \, \partial x'_\nu}, & x'_\mu \\ x'_\nu & , 0 \end{vmatrix} \neq 0 \text{ ***)} \tag{3}$$

*) Es handelt sich hier um den allgemeinen anisotropen aber homogenen Raum [5], [4] und nicht um dessen besonderen Fall der unter demselben Namen in der speziellen Relativitätstheorie behandelt wird.

**) Die griechischen Indexe nehmen die Werte $1, \ldots, n$ an.

***) Wir verwenden hier die Summationskonvention nach wiederholten Indexen.

erfüllt [3]. Der Minkowskische Raum kann analog durch den Ausdruck

(4) $$ds = M(dy_i)\ ^{*)},$$

definiert werden, wo die Funktion $M(y_i')$ die folgenden Bedingungen erfüllt:

$$M(ky_i') = kM(y_i'), \qquad k > 0,$$

$$M_1 = \frac{-1}{(y_i' y_i')^2} \begin{vmatrix} \frac{\partial^2 M}{\partial y_i' \partial y_j'}, & y_i' \\ y_j' & , 0 \end{vmatrix} \neq 0.$$

Sie zeichnet sich aber von der Funktion F dadurch aus, dass sie von den nichtgestrichenen Variabeln y_i unabhängig ist:

(5) $$\frac{\partial M}{\partial y_i} = 0\ ^{**)}.$$

Wenn $m \geqslant n$ ist, so sagen wir, dass der durch die Funktion F bestimmte Finslersche Raum in dem durch die Funktion M definierten Minkowskischen Raum *eingebettet* ist, wenn solche reguläre und analytische Funktionen $\varphi_i(x_\mu)$ existieren, dass

(6) $$M\left(\frac{\partial \varphi_i}{\partial x_\mu} dx_\mu\right) = F(x_\mu, dx_\mu)$$

ist.

Im Jahre 1949 hat W. W. Wagner [6] das oben formulierte Problem (in etwas allgemeinerer Fassung) im bejahender Sinne gelöst und mit Hilfe der Methoden der Theorie der Systeme der partiellen Differentialgleichungen bewiesen, dass $m = 2n$ ist (d. h., dass für einen beliebigen n-dimensionalen Finslerschen Raum ein $2n$-dimensionaler Minkowskischer Raum mit den geforderten Eigenschaften immer existiert, dass es aber einen solchen $(2n-1)$-dimensionalen Minkowskischen Raum nicht immer gibt). Da aber Wagner sich bloss auf die aus der angeführten Theorie bekannten Sätze über die Existenz der Lösungen stützte, so gibt er keine Methode an, die es gestattet, diese Lösungen (d. h. die Funktionen M und φ_i bei vorgegebener Funktion F) explizit anzugeben. Die vorliegende Arbeit stellt sich daher die Aufgabe, eine solche Methode zu finden. Diese Methode wurde dank der oben angegebenen Formulierung des Problems, die viel einfacher ist, als die in der Note [2] mitgeteilte, gefunden. In der Note [2] wurde

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*) Die lateinischen Indexe nehmen die Werte $1, \ldots, n$ an.

**) Die Eigenschaft (5) tritt nur in speziellen Koordinatensystemen auf (was in der nichtkovarianten Schreibweise sichtbar ist), die für Minkowskische Räume immer existieren. Im Falle beliebiger Koordinatensysteme zeichnet sich der Minkowskische Raum unter den Finslerschen Räumen dadurch aus, dass der Krümmungstensor R_{ijkh} und die kovarianten Ableitungen des Torsionstensors $A_{ijk;h}$ verschwinden [4].

das Problem in einer spezielleren Gestalt angegeben, die sich viel zu eng an das bekannte Problem der Einbettung eines Riemannschen Raumes in einem euklidischen Raume anschloss *).

Die gesuchte (spezielle) Lösung erhalten wir, indem wir

$$M(y'_i) = F\left(\frac{y'_1}{y'_{n+1}}, \frac{y'_2}{y'_{n+2}}, \ldots, \frac{y'_n}{y'_{2n}}, y'_{n+1}, \ldots, y'_{2n}\right) \tag{7}$$

und

$$\varphi_1 = \frac{x_1^2}{2}, \varphi_2 = \frac{x_2^2}{2}, \ldots, \varphi_n = \frac{x_n^2}{2}, \varphi_{n+1} = x_1, \ldots, \varphi_{2n} = x_n \tag{8}$$

setzen. Es kann leicht festgestellt werden, dass die so bestimmte Lösung alle angeführten Bedingungen erfüllt. Gegebenenfalls muss man aber, um die Erfüllung der Bedingung (3) zu erreichen, eventuell den Variabilitätsbereich der Variabeln y_i in entsprechender Weise einschränken (was den Beschränkungen auf die Richtungen entspricht, für die das Variationsproblem $\delta \int ds = 0$ regulär ist). Dies wäre nur dann unmöglich (d. h. wir würden durch eine derartige Beschränkung überhaupt alle Richtungen ausschliessen), wenn die Funktion M in allen ihren Argumenten linear wäre (M_1 wäre dann identisch Null), was aber durch (7) und (3) ausgeschlossen ist.

Wir sehen, dass in der gefundenen Lösung $m = 2n$ ist (es scheint, dass die angegebene Lösung uns eine tiefere Begründung für das Bestehen des Wagnerschen Satzes vermittelt). Wir erkennen ausserdem, dass wir, bei einem bestimmten n, für verschiedene F auch verschiedene M, dagegen dieselbe Einbettungsweise**) finden. Die Finslersche Geometrie ist mit der inneren Geometrie einer algebraischen Hyperfläche zweiten Grades (vom parabolischen Typus) in einem Minkowskischen Raume identisch. Bei den angegebenen Lösungen ist die Situation gerade umgekehrt, als im Falle der Einbettung eines Riemannschen Raumes in einem euklidischen Raum, wo der Raum, in dem die Einbettung stattfindet, (beim gegebenen n) immer derselbe ist, dagegen die Art der Einbettung (die Hyperfläche) mit der Änderung des eingebetteten Raumes variiert.

Der Verfasser drückt A. Krzywicki, Mitarbeiter der Abteilung für geometrische Optik, für die Diskussion und die Mitwirkung an der Lösung des Problems seinen besten Dank aus.

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*) In der Note [2] wurde dabei ein terminologischer Fehler begangen. In Anlehnung an Cartan [4] hat der Verfasser behauptet, dass die Metrik des eingebetteten Finslerschen Raumes durch die Metrik des Minkowskischen Raumes „induziert" („*induite*") wird und dass sie für die betreffende Hyperfläche nicht notwendig „*intrinsèque*" ist. Diese beiden Ausdrücke sind zu vertauschen; ausserdem beziehen sie sich bei Cartan auf den Zusammenhang (*connexion*) des Raumes und nicht auf die Metrik desselben (auf den metrischen Tensor).

**) d. h. dieselben Funktionen $\varphi_i(x_\mu)$.

SCHRIFTTUM

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POLONAISE DES SCIENCES
Cl. III — Vol. II, No. 7, 1954

MATHÉMATIQUE

Sur l'équation généralisée des oscillations entretenues

par

Z. MIKOŁAJSKA

Présenté par T. WAŻEWSKI le 24 Avril 1954

On a consacré de nombreux travaux à l'étude des solutions périodiques de l'équation $\ddot{x} = g(x) + \dot{x} \cdot f(x, \dot{x})$. Dans la présente note nous remplaçons $g(x)$ par $g(x, \dot{x})$.

Le théorème 1 a un caractère bien général. Le théorème 2 présente son application au cas particulier analogue à un théorème de Levinson et Smith [1] et de Adamoff [2].

I. Envisageons les équations

$$\ddot{x} = g(x, \dot{x}), \tag{1}$$

$$\ddot{x} = g(x, \dot{x}) + p(x, \dot{x}), \tag{2}$$

ou, ce qui revient au même, les deux systèmes d'équations

$$\dot{x} = v, \quad \dot{v} = g(x, v), \tag{1 a}$$

$$\dot{x} = v, \quad \dot{v} = g(x, v) + p(x, v), \tag{2 a}$$

les fonctions g et p étant partout continues.

Relativement à (1 a) et à (2 a) nous admettons les hypothèses suivantes:

H_1. L'origine (0,0) constitute l'unique point singulier de (1 a) et de (2 a).

H_2. Toutes les solutions de (1 a) sont périodiques.

H_3. Le système (1 a) admet une intégrale première $W(x, v)$ continue à l'origine, de classe C^1 partout ailleurs; telle que

$$W_v(x, v) > 0 \quad \text{pour} \quad v > 0, \qquad W_v < 0 \quad \text{pour} \quad v < 0, \quad W(0, 0) = 0 \tag{3}$$

et telle que l'on a la convergence uniforme

$$\begin{aligned} W_x \Rightarrow 0 \quad \text{pour} \quad |v| \to \infty, \qquad W_v \Rightarrow 1 \quad \text{pour} \quad v \to +\infty, \\ W_v \Rightarrow -1 \quad \text{pour} \quad v \to -\infty, \end{aligned} \tag{4}$$

lorsque x varie dans l'intervalle $\Delta = [-x_1, x_1]$ où x_1 désigne un certain nombre positif.

(5) $$vp(x,v) \leqslant 0, \quad \text{c'est-à-dire} \quad W_v p \leqslant 0 \quad \text{pour} \quad |x| \geqslant x_1.$$

H_4. Il existe un $\eta > 0$ fixe et deux fonctions $a(x)$ et $b(x)$ intégrables au sens de Lebesgue dans Δ telles que

(6) $$\begin{aligned} p(x,v) &\leqslant v\,a(x) \quad \text{pour} \quad |x| \leqslant x_1, \qquad v \geqslant \eta; \\ p(x,v) &\geqslant -v\,b(x) \quad \text{pour} \quad |x| \leqslant x_1, \qquad v \leqslant -\eta; \end{aligned}$$

(7) $$\int_{-x_1}^{x_1} a(x)\,dx < \int_{-x_1}^{x_1} b(x)\,dx.$$

H_5. En introduisant les notations:

(8) *courbe* $C(k)$: $W(x,v) = k$, *domaine* $D(k)$: $W(x,v) \leqslant k$, nous admettons que dans un voisinage d'une courbe $C(k_0)$ $(k_0 > 0)$ on ait

(9) $$vp(x,v) \geqslant 0, \qquad \text{c'est-à-dire} \quad W_v p \geqslant 0 \ \text{(cf. (3))}.$$

Théorème 1. *Dans les hypothèses H_1-H_5:*

1° *La classe de toutes les intégrales périodiques du système* (2 a) *est vide ou représente un ensemble borné.*

2° *Le système* (2 a) *admet au moins une intégrale périodique. (Il s'agit des intégrales ne se réduisant pas à un seul point).*

Démonstration. I. Observons que les courbes $C(k)$ $(k > 0)$ constituent toutes les solutions périodiques du système (1 a).

Il existe un $\delta > 0$ tel que (cf. (7))

(3,1) $$\int_{-x_1}^{x_1} [a(x) + 2\delta]\,dx < \int_{-x_1}^{x_1} [b(x) - 2\delta]\,dx.$$

Soit $v_1 > \eta$. Nous definirons une courbe simple fermée de Jordan $J(v_1)$ constituant la frontière d'un domaine borné $K(v_1)$, telle qu'aucune intégrale de (2 a) issue de $K(v_1)$ ne pourra quitter $K(v_1)$. Elle sera composée de cinq arcs simples reliant les points $P_1, \ldots, P_5$, définis par les conditions suivantes:

$$\widehat{P_1P_2}: v = \varphi(x) = v_1 + \int_{-x_1}^{x} [a(s) + \delta]\,ds, \quad |x| \leqslant x_1,$$

$$\widehat{P_2P_3}: W(x,v) = W(P_2), \quad x \geqslant x_1.$$

Les coordonnées de P_3 seront désignées par (x_1, v_3), $(v_3 < 0$ lorsque v_1 est suffisamment grand).

$$\widehat{P_3P_4}: v = \psi(x) = v_3 - \int_{x_1}^{x} [b(s) - \delta]\,ds \quad |x| \leqslant x_1$$

$$\widehat{P_4P_5}: W(x,v) = W(P_4), \quad x \leqslant -x_1$$

$\widehat{P_5P_1}$: segment rectiligne aux extrémités P_5 et P_1

II. P_1 est situé au dessus de P_5, c'est-à-dire $W(P_1) > W(P_5)$ (cf. (3)), pour v_1 suffisamment grand. Nous avons, en effet, $W(P_1) - W(P_5) = W(P_1) - W(P_2) + W(P_3) - W(P_4)$, car $W(P_2) = W(P_3)$, $W(P_4) = W(P_5)$.

On a, suivant le théorème sur les accroissements finis:

$$W(P_2) - W(P_1) = W(x_1, \varphi(x_1)) - W(-x_1, \varphi(-x_1)) =$$
$$= 2x_1 W_x(Q_1) + [\varphi(x_1) - \varphi(-x_1)]\,[W_v(Q_1) - 1] + \varphi(x_1) - \varphi(-x_1).$$

Or pour $v_1 \to +\infty$ la distance minima de $J(v_1)$, à l'origine, tend vers $+\infty$. En vertu de (4) on a pour v_1 suffisamment grand

$$W(P_2) - W(P_1) < \varphi(x_1) - \varphi(-x_1) + 2x_1\delta = \int_{-x_1}^{x_1} [a(s) + 2\delta]\, ds.$$

D'une façon analogue, on obtient

$$W(P_3) - W(P_4) > \int_{-x_1}^{x_1} [b(s) - 2\delta]\, ds.$$

En ajoutant ces inégalités on voit que $W(P_1) > W(P_5)$.

III. Pour v_1 suffisamment grand, les intégrales de (2 a) ne peuvent sortir de $K(v_1)$ par aucun des cinq arcs formant la frontière de $K(v_1)$. C'est évident pour les points de $\widehat{P_5 P_1}$, différents de P_1 (cf. II), c'est-à-dire pour les points appartenant à $\widehat{P_5 P_1} - P_1$. Il en est de même pour les points de $\widehat{P_1 P_2} - P_2$. Soit $B(t) = (x(t), v(t))$ une intégrale quelconque de (2 a) envisagée dans un intervalle $S(h): t_0 \leqslant t \leqslant t_0 + h$ suffisamment petit. Soit $B(t_0) \in \widehat{P_1 P_2} - P_2$. On a presque partout dans $S(h)$

$$\sigma'(t) = \frac{d}{dt}\Big(v(t) - \varphi(x(t))\Big) = \left(\frac{g}{v} + \frac{p}{v} - a(x) - \delta\right) v.$$

Or, W étant l'intégrale première de (1 a) on a, en vertu de (4):

$$\frac{g}{v} = -\frac{W_x}{W_v} \Rightarrow 0 \quad \text{dans } \Delta \quad \text{pour} \quad |v| \to \infty.$$

Il en résulte en vertu de (6) que, pour v_1 suffisamment grand, on a presque partout dans $S(h): \sigma'(t) \leqslant 0$ et, par suite, $v(t) - \varphi(x(t)) \leqslant v(t_0) - \varphi(x(t_0)) = 0$, ce qui signifie que, pour $t \in S(h)$, $B(t)$ parcourt au-dessous de $\widehat{P_1 P_2}$, donc dans $K(v_1)$. D'une façon analogue, on démontre que des intégrales de (2 a) ne peuvent pas sortir de $K(v_1)$ en traversant $\widehat{P_3 P_4}$.

Soit $B(t_0) \in P_2 P_3 - P_3$. Comme W est une intégrale première de (1 a) on a dans $S(h)$ (cf. (5))

$$\frac{d}{dt} W(x(t), v(t)) = W_x v + W_v g + W_v p = W_v p \leqslant 0.$$

Donc $W(x(t), v(t)) \leqslant W(B(t_0))$ pour $t \in S(h)$, ce qui signifie que pour $t \in S(h)$ l'intégrale $B(t)$ parcourt dans $D(W(B(t_0)))$ (cf. (8)) et, à plus forte raison, dans $K(v_1)$. On démontre pareillement que, pour v_1 suffisamment grand, l'intégrale de (2 a) issue de P_1 parcourt toujours dans $K(v_1)$. Elle ne pourra donc pas revenir au point P_1, car P_5 est au-dessous de P_1. La première partie du théorème se trouve donc établie.

En vertu de l'hypothèse H_5, les intégrales de (2 a) ne peuvent pas sortir du domaine fermé dont la frontière est composée de $C(k_0)$ et de $J(v_1)$ (v_1 suffisamment grand), lequel domaine ne contient aucun point singulier. La seconde partie de notre théorème résulte donc d'un théorème de Poincaré-Bendixson [3].

Théorème 2. *Relativement à* (1 a) *et* (2 a) *nous admettons les hypothèses suivantes:*

(K_1) $g(x,v)$ *est une fonction de classe* C^1,

$g(x,0) > 0$ *pour* $x < 0$,

$g(x,0) < 0$ *pour* $x > 0$,

$g(x,v) \equiv g(x,-v)$, $p(x,v) = f(x,v)\,v$.

(K_2) *Il existe des fonctions* $\Phi_1(x)$, $\Phi_2(x)$ *continues partout et des fonctions* $\Psi_1(u)$, $\Psi_2(u)$ *continues et positives pour* $u \geqslant 0$, *telles que*

$$\Phi_2(x)\cdot\Psi_2(v^2) \leqslant g(x,v) \leqslant \Phi_1(x)\,\Psi_1(v^2) \quad \textit{pour} \quad v > 0,$$

$$\int_0^{+\infty}\Phi_1(x)\,dx = \int_0^{-\infty}\Phi_2(x)\,dx = -\infty,$$

$$\int_0^{+\infty}\frac{du}{\Psi_i(u)} = +\infty \quad (i = 1, 2).$$

(K_3) *Il existe un* $x_0 > 0$ *tel que* $f(x,v) \leqslant 0$ *pour* $|x| \geqslant x_0$, *et deux nombres finis* $M \geqslant 0$, $\eta \geqslant 0$ *tels que* $f(x,v) \leqslant M$ *pour* $|x| \leqslant x_0$, $|v| \geqslant \eta$.

(K_4) *Il existe* $x_1 > x_0$ (x_1 *fixe*) *et une fonction* $a^*(x)$ *intégrable au sens de Lebesgue pour* $x_0 \leqslant x \leqslant x_1$ *et telle que*

$$f(x,v) \leqslant a^*(x) \quad \textit{pour} \quad x_0 \leqslant x \leqslant x_1,\ \ v \leqslant \eta, \quad \textit{et} \quad \int_{x_0}^{x_1} a^*(x)\,dx < -4Mx_0.$$

(K_5) *Dans l'intervalle* $[-x_1, x_1]$ *a lieu la convergence uniforme*

$$\frac{g(x,v)}{v} \Rightarrow 0 \quad \textit{pour} \quad |v| \to +\infty, \quad \textit{et} \quad \frac{g_v(x,v)}{v} \Rightarrow 0 \quad \textit{pour} \quad |v| \to +\infty.$$

(K_6) $f(0,0) > 0$.

Dans ces hypothèses l'équation (2) (*ou* (2 a)) *a les propriétés* 1° *et* 2° *du théorème 1.*

En effet, on définit W par les conditions $vW_x + gW_v \equiv 0$, $W(0,v) \equiv |v|$. a et b (cf. H_4) peuvent être définis par les relations suivantes:

$$a(x) = M \quad \text{pour} \quad -x_0 \leqslant x \leqslant x_0; \qquad a(x) = 0 \quad \text{pour} \quad -x_1 \leqslant x \leqslant -x_0;$$
$$a(x) = a^*(x) \quad \text{pour} \quad x_0 \leqslant x \leqslant x_1; \quad b(x) = -M \quad \text{pour} \quad -x_0 \leqslant x \leqslant x_0;$$
$$b(x) = 0 \quad \text{pour} \quad x_0 \leqslant |x| \leqslant x_1.$$

D'après (K_3) et (K_4): $p = vf \leqslant 0 \cdot v = av$ pour $v \geqslant \eta$, $-x_1 \leqslant x \leqslant -x_0$; $fv \leqslant Mv = av$ pour $v \geqslant \eta$, $x_0 \geqslant |x|$;

$$fv \leqslant a^* v = av \quad \text{pour} \quad v \geqslant \eta,\ x_0 \leqslant x \leqslant x_1,$$

D'une façon analogue

$$vf \geqslant Mv = -bv \text{ pour } v \leqslant -\eta,\ -x_0 \leqslant x \leqslant x_0,\ vf \geqslant 0 = -vb \text{ pour } v \geqslant -\eta,$$
$$x_0 \leqslant |x| \leqslant x_1.$$

$$\int_{-x_1}^{x_1} a(x)\,dx = 2Mx_0 + \int_{x_0}^{x_1} a^*(x)\,dx < -2Mx_0 = \int_{-x_1}^{x_1} b(x)\,dx.$$

L'hypothèse H_1 résulte de (K_1); H_2 résulte de (K_1) et (K_2); H_3 de (K_1), (K_2), (K_3), et (K_5); H_5 résulte de (K_6).

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. II, No. 7, 1954

MATHÉMATIQUE

Remarques sur un théorème de T. Ważewski relatif à l'allure asymptotique des intégrales des équations différentielles

par
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Présenté par T. WAŻEWSKI le 24 Avril 1954

1. Soit

$$\frac{dx_i}{dt} = f_i(t, x_1, \ldots, x_n), \qquad i = 1, 2, \ldots, n \tag{1}$$

un système d'équations différentielles, les fonctions f_i étant définies et continues dans un ensemble ouvert D de l'espace euclidien R^{n+1} à $n+1$ dimensions. Nous supposerons que par chaque point de D il passe une intégrale et une seule du système (1).

Soient $x_i = x_i(t, p)$ les équations de l'intégrale du système (1) qui passe par le point $p \in D$; posons

$$\varphi(p, t) = (t, x_1(t, p), \ldots, x_n(t, p)) \in R^{n+1}.$$

Nous désignerons par t_p la projection du point $p \in R^{n+1}$ sur l'axe t:

$$t_p = \mathrm{pr}_t(p)$$

et par $\mathrm{pr}_x(p)$ la projection de p sur l'hyperplan $R^n(x_1, \ldots, x_n)$.

On sait [3] que pour tout $p \in D$, il existe un intervalle ouvert maximal (borné ou non) $\Delta(p) \subset R$, tel que $\varphi(p, t) \in D$, à condition que $t \in \Delta(p)$.

Les ensembles

$$L(p) = \{\varphi(p, t) \mid t \in \Delta(p)\},$$
$$L^+(p) = \{\varphi(p, t) \mid t \in \Delta(p), t \geqslant t_p\},$$
$$L^-(p) = \{\varphi(p, t) \mid t \in \Delta(p), t \leqslant t_p\},$$

seront nommés respectivement l'intégrale, la demi-intégrale droite et la demi-intégrale gauche qui passe par le point $p \in D$.

Soit G un ensemble ouvert, $G \subset D$. Désignons par $\mathrm{Fr}(G)$ et $\mathrm{Ext}(G)$ respectivement la frontière et l'exterieur de l'ensemble G. Avec T. Ważewski [5], nous dirons qu'un point $p \in \mathrm{Fr}(G) \cap D$ est un point de sortie de G

relatif au système (1), s'il existe un nombre $\varepsilon > 0$ tel que $\varphi(p, t) \in G$ pour $t_p - \varepsilon < t < t_p$. Lorsqu'il existe un $\varepsilon > 0$ tel que

$$\varphi(p, t) \in G \qquad \text{pour} \quad t_p - \varepsilon \leqslant t < t_p,$$

$$\varphi(p, t) \in \mathrm{Ext}(G) \quad \text{pour} \quad t_p < t \leqslant t_p + \varepsilon,$$

le point $p \in \mathrm{Fr}(G) \cap D$ s'appelle point de sortie stricte de G relatif au système (1).

Ważewski a démontré [5] que si l'ensemble S_0 des points de sortie d'un ensemble ouvert $G \subset D$ relatifs au système (1) coïncide avec l'ensemble S des points de sortie stricte et s'il existe deux ensembles Z et S_1, ayant les propriétés:

a) $S_1 \subset S$, $Z \subset G \cup S_1$, $Z \cap G \neq 0$, $Z \cap S_1 \neq 0$;

b) $Z \cap S_1$ est un rétracte de S_1;

c) $Z \cap S_1$ n'est pas un rétracte de Z,

alors il existe un point $p_0 \in Z \cap G$, tel que: ou bien $L^+(p_0) \subset G$, ou bien le premier point d'intersection de $L^+(p_0)$ avec $\mathrm{Fr}(G)$ appartient à $S - S_1$.

2. En utilisant les idées et les résultats de T. Ważewski, ainsi que la notion de rétracte par déformation *), nous démontrons dans cette note trois théorèmes, le dernier permettant d'établir dans certaines conditions l'existence de solutions périodiques pour un système dynamique de deux équations différentielles.

Rappelons qu'un sous-espace A d'un espace topologique E est dit un rétracte de E, lorsqu'il existe une transformation continue $r: E \rightarrow A$, telle que $r(a) = a$ pour $a \in A$; la transformation r est dite rétraction [1].

Un sous-espace A d'un espace topologique E est dit un rétracte par déformation de E, lorsqu'il existe une rétraction $r: E \rightarrow A$, homotope à l'identité [2], c'est-à-dire s'il existe une transformation continue $\alpha: E \times I \rightarrow E$, où I désigne l'intervalle fermé $[0,1]$, telle que

$$\alpha(x, 0) = x, \qquad \alpha(x, 1) = r(x).$$

Théorème 1. *Soit $G \subset D$ un ensemble ouvert et S l'ensemble des points de sortie stricte de G relatifs au système (1). Si chaque point de sortie de G relatif au système (1) est un point de sortie stricte et s'il existe un ensemble $S_1 \subset S$ et un ensemble fermé Z ayant les propriétes suivantes*:

a) $Z \subset G \cup S_1$, $Z \cap G \neq 0$, $Z \cap S_1 \neq 0$,

b) $Z \cap S_1$ *est un rétracte de* $\bar{S}_1$,

c) $Z \cup \bar{S}_1$ *est un rétracte de* $\bar{G}$,

d) $Z \cap S_1$ *n'est pas un rétracte par déformation de* Z,

alors il existe un point $p_0 \in Z \cap G$ tel que: ou bien $L^+(p_0) \subset G$, ou bien le premier point d'intersection de $L^+(p_0)$ avec $\mathrm{Fr}(G)$ appartient à $S - S_1$.

*) Je dois la suggestion d'utiliser cette notion à T. Ganea.

Démonstration. Admettons que pour tout $p \in Z \cap G$, la demi-intégrale $L^+(p)$ ait un premier point d'intersection avec $\mathrm{Fr}(G)$, appartenant à S_1. Considérons la transformation $h: Z \to S_1$, définie de la façon suivante:

si $p \in Z \cap G$, $h(p)$ est le premier point d'intersection de $L^+(p)$ avec $\mathrm{Fr}(G)$,
si $p \in Z \cap S_1$, $h(p) = p$.

Cette transformation est continue [5]. En désignant par r la rétraction de $\bar{S}_1$ en $Z \cap S_1$, on voit que la transformation rh est une rétraction de Z en $Z \cap S_1$. Nous allons prouver que rh est homotope à la transformation identique de Z en Z.

Définissons une transformation $g: Z \cup \bar{S}_1 \to Z$, en posant

$$g(p) = p \quad \text{pour} \quad p \in Z,$$
$$g(p) = r(p) \quad \text{pour} \quad p \in \bar{S}_1.$$

Comme $r(p) = p$ pour $p \in Z \cap \bar{S}_1 = Z \cap S_1$, et les ensembles Z et $\bar{S}_1$ sont fermés, la transformation g est continue.

Soit maintenant $\tau_p = \mathrm{pr}_t\, h(p)$; pour tout point $(p, s) \in Z \times I$, on a $\varphi(p, t_p(1-s) + \tau_p \cdot s) \in \bar{G}$, car $t_p \leqslant t_p(1-s) + \tau_p \cdot s \leqslant \tau_p$. En désignant par ϱ la rétraction de $\bar{G}$ en $Z \cup \bar{S}_1$, définissons une transformation $\alpha: Z \times I \to Z$ de la façon suivante:

$$\alpha(p, s) = g\varrho\varphi(p, t_p(1-s) + \tau_p \cdot s).$$

Il est évident que la transformation α est continue et que

$$\alpha(p, 0) = p, \ \alpha(p, 1) = rh(p),$$

d'où résulte la contradiction qui démontre le théorème.

Remarque 1. On voit facilement que si $S_1 = S$, le théorème 1 donne une condition pour l'existence d'une demi-intégrale $L^+(p_0) \subset G$.

3. Théorème 2. *En appliquant le théorème 1, ou bien le théorème de T. Ważewski, au cas où le système (1) est dynamique (c. à d. au cas où les fonctions f_i ne dépendent pas de t) on peut, sans changer les énoncés formels de ces théorèmes, se borner à la considération de l'espace des phases composé des points $(x_1, \ldots, x_n)$.*

Nous nous dispensons de la démonstration de ce théorème.

4. Soit

$$\frac{dx}{dt} = X(x, y), \qquad \frac{dy}{dt} = Y(x, y) \tag{2}$$

un système dynamique défini dans un ensemble ouvert H appartenant au plan à deux dimensions R^2.

Théorème 3. *Soient Γ_1 et Γ_2 deux courbes simples fermées dans R^2, Γ_2 étant contenue dans l'intérieur de Γ_1 et soit G le domaine borné par Γ_1 et Γ_2 tel que $\bar{G} \subset H$. Si l'ensemble des points de sortie de G coïncide*

avec l'ensemble S des points de sortie stricte et si $\Gamma_1 \neq \bar{S} \neq \Gamma_2$, alors le système admet au moins une solution périodique pouvant se réduire à un point singulier qui appartient à l'ensemble fermé $\bar{G}$).*

Démonstration. Si $S = 0$, l'affirmation résulte d'un théorème de Poincaré-Bendixson ([3], p. 223). Soit donc $S \neq 0$; il y a deux cas possibles:

I. L'ensemble S n'est pas connexe. Soient C_1 et C_2 deux composantes de S et soit $a_1 \in C_1$, $a_2 \in C_2$. Considérons un arc simple Z, unissant les points a_1 et a_2, tel que $Z \subset G \cup S$, $Z \cap S = \{a_1, a_2\}$. On voit facilement que $Z \cap S$ est un rétracte de S et n'est pas un rétracte de Z. En tenant compte du théorème 2, on peut appliquer le théorème de Ważewski, d'où résulte l'existence d'une demi-trajectoire $L^+(p) \subset G$. Comme $\bar{G}$ est borné, le système (4) admet au moins une solution périodique appartenant à $\bar{G}$.

II. L'ensemble S est connexe. Comme $\Gamma_1 \neq \bar{S} \neq \Gamma_2$, l'ensemble $\bar{S}$ est un arc simple. Soient $a \in S$ et Z une courbe simple fermée qui passe par le point a et dont l'intérieur contient l'intérieur de Γ_2, $Z \subset G \cup S$, $Z \cap S = \{a\}$. On voit que l'ensemble Z satisfait aux conditions du théorème 2, avec $S_1 = S$; un raisonnement identique à celui de I prouve l'existence d'une solution périodique pour le système (2).

Remarque 2. En appliquant la substitution $t = -T$, on obtient trois théorèmes analogues relatifs aux demi-intégrales gauches.

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*) L'idée d'appliquer le théorème 1, ainsi que le théorème de T. Ważewski aux systèmes dynamiques dans R^3, m'a été suggérée par A. Halanay.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. II, No. 7, 1954

MATHÉMATIQUE

Sur le théorème de Gödel pour les théories indénombrables

par

J. ŁOŚ

Présenté par K. KURATOWSKI le 29 Avril 1954

Le théorème de Gödel sur l'existence des modèles des systèmes non-contradictoires a été dernièrement formulé et démontré non seulement pour les théories dites dénombrables, mais aussi pour les théories indénombrables, c. à d. pour les théories dans lesquelles apparaît une infinité indénombrable de signes [1], [2], [5].

Cette formulation est certainement plus générale et elle admet plus d'applications. Le théorème de Gödel dans cette formulation cesse d'être un théorème purement logique et devient un moyen de démonstrations mathématiques. Notons que ce moyen n'est pas effectif: j'ai démontré [2] que le théorème de Gödel pour les théories indénombrables implique le principe d'ordre (chaque ensemble peut être ordonné).

Le but de cette note est de démontrer que le théorème de Gödel pour les théories indénombrables équivaut effectivement aux théorèmes d'existence des idéaux premiers dans les algèbres de Boole (voir p. ex. [3], proposition (1)).

Ce dernier théorème a été démontré par plusieurs auteurs (S. Ulam, A. Tarski, M. H. Stone), premièrement pour les besoins de la théorie de la mesure, secondement comme lemme principal dans la théorie de la représentation des algèbres de Boole.

Les dernières recherches sur le théorème de Gödel prouvent que le théorème d'existence des idéaux premiers, appliqué à l'algèbre de Boole, formé de propositions de la théorie envisagée, suffit pour démontrer le théorème de Gödel, au moins dans le cas des théories ouvertes, c. à d. pour les théories dans les propositions desquelles n'apparaissent pas les quantificateurs (on constate ce fait facilement en examinant les démonstrations proposées dans les travaux de Rasiowa-Sikorski [4] ou de Łoś [2]).

Or, le passage aux théories avec les quantificateurs peut être effectué par la méthode d'élimination des quantificateurs, sans l'usage de l'axiome du choix.

Il ne reste alors que de démontrer que le théorème de Gödel implique le théorème d'existence des idéaux.

Soit $\boldsymbol{B}$ une algèbre de Boole. Supposons que pour chaque élément $a \in \boldsymbol{B}$ est donné un "signe" f_a, de sorte que la fonction $f(a) = f_a$ est biunivoque.

Désignons par $\mathfrak{X}$ l'ensemble formé des axiomes de l'algèbre de Boole, de trois propositions suivantes

$$\varphi(x+y) = \varphi(x) + \varphi(y)$$
$$\varphi(x') = \varphi(x)'$$
$$\varphi(x) = f_0 \vee \varphi(x) = f_1$$

(φ étant un signe de fonction; 0 et 1 les éléments minimal et maximal de $\boldsymbol{B}$), et de toutes les propositions qui forment la "description" de l'algèbre $\boldsymbol{B}$, c. à d. des propositions de la forme:

$$f_a + f_b = f_{a+b}$$
$$f_{a'} = f'_a.$$

Puisque chaque algèbre de Boole finie admet un homomorphisme sur l'algèbre formée de deux éléments (0,1), il résulte que chaque sous-ensemble fini de $\mathfrak{X}$ n'est pas contradictoire, et par conséquent que l'ensemble $\mathfrak{X}$ entier n'est pas contradictoire. D'après le théorème de Gödel, il existe un modèle

$$A, +, ', \Phi, \{F_a\}_{a \in B}$$

de $\mathfrak{X}$ (Φ est une interprétation pour φ et F_a pour f_a). On constate facilement que l'ensemble

$$\underset{a \in \boldsymbol{B}}{E}[\Phi(F_a) = F_0]$$

est un idéal premier de l'algèbre $\boldsymbol{B}$, c. q. f. d.

Le théorème d'existence des idéaux premiers dans les algèbres de Boole, et ce qui en suit, le théorème de Gödel aussi, équivaut effectivement à plusieurs théorèmes [3], p. ex. au théorème de Tychonoff sur le produit des espaces bicompacts (le produit cartésien des espaces bicompacts est un espace bicompact). On ne sait pas s'il équivaut au principe du choix.

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BULLETIN DE L'ACADÉMIE
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Cl. III — Vol. II, No. 7, 1954

MATHEMATICS

A Topological Characterization of 2-Polytopes

by

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Presented by K. BORSUK on May 6, 1954

1.1. Various topological characterizations of 2-sphere, 2-cell and 2-manifolds have been known for a long time (for instance [1]). It is the aim of this note to give a set of topological properties which characterize 2-polytopes among the metric and separable spaces. The knowledge of such a characterization — independently of its methodological value — enables us to solve some problems in the theory of Cartesian products [4].

1.2. All spaces under consideration will be presupposed separable and metric. By a polytope we shall always mean a compact subset of Euclidean n-space homeomorphic to a Euclidean polytope in the sense of [2] p. 128—129. A polytope of dimension $\leqslant 1$ will be called a graph. The topological characterization of graphs was known long ago.

If K is any topological space, then by $\mathrm{reg}_n K$ we shall denote the set of n-regular points of K, i. e. the set of points having a neighbourhood homeomorphic to Euclidean n-space.

In particular, each component of the set $\mathrm{reg}_2 K$ is an (open or closed) 2-manifold, hence its topological characterization is known.

We shall use the abbreviations ANR = a compact absolute neighbourhood retract (see [3]); almost all = all save possibly a finite number.

2.1. Let K be a 2-dimensional ANR. Four cases are possible:

a) $\dim(K - \mathrm{reg}_2 K) = 2$, b) $\dim(K - \mathrm{reg}_2 K) = 1$,
c) $\dim(K - \mathrm{reg}_2 K) = 0$, d) $\dim(K - \mathrm{reg}_2 K) = -1$.

In case a), K obviously is not a polytope. The ANR's satisfying a) exist really: denote by Q_n an n-cell and by P a 1-dimensional ANR in which the points of order $\neq 2$ are everywhere dense. Then $K_n = Q_{n-1} \times P$ is an n-dimensional ANR and $\mathrm{reg}_i K_n = 0$ for each i.

Evidently condition b) is insufficient to insure that K be a polytope. It is interesting to note that even the much stronger condition requiring $K - \mathrm{reg}_2 K$ to be a graph is insufficient (see example in 2.3).

In case c), the following theorem holds:

Theorem 1. *Let K be an at most 2-dimensional ANR. If* $\dim(K - \mathrm{reg}_2 K) \leqslant 0$, *then K is a polytope*).*

Theorem 1 cannot be generalized to *ANR*'s of dimension > 2. For let us consider the following example: let L_k, $k = 1, 2, \ldots$, be a sequence of disjoint Antoine arcs lying in a 3-sphere S_3 and converging to a point $a_0 \in S_3$. The set S^* obtained from S_3 by pinching each arc L_k to a point a_k is a 3-dimensional *ANR*. But $S^* - \mathrm{reg}_3 S^* = \{a_0, a_1, \ldots\}$.

2.2. From theorem 1 we have a corollary which gives us the topological characterization of 2-dimensional closed pseudomanifolds. (For the definition of closed pseudomanifolds see [2], p. 193. We give a characterization among the continua and precisely this is essential. The characterization of closed pseudomanifolds among the polytopes is known).

Corollary. In order that a 2-dimensional continuum be a closed pseudomanifold it is necessary and sufficient that it be an *ANR* in which the set of 2-regular points is connected and its complement at most 0-dimensional.

Proof. Suppose the continuum K fulfills the conditions above. By th. 1, K is a polytope, hence the set $K - \mathrm{reg}_2 K$ is finite. The connectedness of K insures that it is contained in a closure of $\mathrm{reg}_2 K$. Consequently K is a homogeneously 2-dimensional polytope in which the set of 2-regular points is connected and its complement finite. By [2] p. 403, K is a closed pseudomanifold. Thus the sufficiency is established. The necessity is known (see [2] l. c.).

2.3. Theorem 1 is needed in the proof of the following theorem 2, which gives a topological characterization of 2-polytopes.

Theorem 2. *In order that a (separable, metric) space K be a polytope of dimension $\leqslant 2$, it is necessary and sufficient that*

1° *K be an ANR,*

2° *$K = A + B$, where $A \subset \mathrm{reg}_2 K$ and B is a graph,*

3° *almost all points $p \in K$ have arbitrarily small neighbourhoods U such that for each component S of $U - B$ the set $S + U \cdot B - (p)$ is connected*).*

Conditions 1° and 2° need no explanation. Condition 3° restricts the position of graph B in K. It may be replaced by either of the following two:

3°′ Almost all points $p \in K$ satisfy the following proposition: given $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in A$, $y \in B$ and $\varrho(p, x) + \varrho(p, y) < \delta$ then there exists in $A + (y)$ an arc $L = xy$ of diameter less than ε.

3°″ Almost all points $p \in K$ have arbitrarly small neighbourhoods U such that if $q \in U \cdot B$, then $Fr(U)$ is a deformation retract of $U - (q)$. (Fr denotes boundary).

*) The proof will appear in Fundamenta Mathematicae.

Condition 3^0 is essential, as the following example shows: Let Q be a square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. Consider the upper semicontinuous decomposition of Q, of which the elements are:

a) pairs $[(1/n, y), (y+1/n, 0)]$ where $0 \leqslant y \leqslant 1/2^n$, $n = 2, 3, 4, \ldots$

b) all other individual points of Q.

Denote by Q^* the hyperspace of this decomposition. By th. (T) in [3], Q^* is an *ANR*. It is easy to see that it is possible to decompose Q^* into the sum of two sets, one of which is an open 2-cell and the second a simple closed curve. Hence conditions 1^0 and 2^0 of th. 1 are fulfilled, although obviously Q^* is not a polytope.

3. From theorem 2 there immediately follows:

Theorem 3. *A compact space which is locally a 2-polytope, is a polytope,*

Proof. Since K is locally a polytope (i. e. each points is contained in a neighbourhood homeomorphic to a neighbourhood in a polytope). K is an *ANR*. Denote $A = \mathrm{reg}_2 K$, $B = K - A$. Of course, B is locally a polytope of dimension $\leqslant 1$. Thus B is a graph and the conditions 1^0 and 2^0 from th. 2 are fulfilled. Since a polytope satisfies 3^0, the compactness of K and B insures that K also satisfies it. Hence the theorem is proved.

Although the above theorem can be obtained directly, and some authors mention it as a known theorem, it seems to me that its proof has never been published.

By applying the theorems of G. S. Young [5] one can also obtain a characterization of 2-polytopes which does not explicitly distinguish the set of irregular points.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. II, No. 7, 1954

MATHEMATICS

On 2-Dimensional Topological Divisors of Polytopes

by

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Presented by K. BORSUK on May 6, 1954

A space A is called a topological divisor of a space K if there exists a space B such that the product $A \times B$ is homeomorphic to K. Professor Borsuk raised the following problem: Is a topological divisor of a polytope always a polytope? In this paper we give a partial answer to this problem and investigate some related questions.

Theorem 1. *An at most 2-dimensional topological divisor of a compact polytope is also a polytope.*

Proof. We denote by $\mathrm{reg}_n K$ the set of n-regular points of K, i. e. the set of points of K having a neighbourhood in K homeomorphic to the Euclidean n-space.

Let K be an n-dimensional compact polytope and let $K = A \times B$, where $\dim A \leqslant 2$. We shall show that A is a polytope. If $\dim A = 0$, the theorem is trivial. If $\dim A = 1$, it can be obtained by a simple modification of the proof of 6. in [2]. Hence we may assume that $\dim A = 2$.

Denote:

$$K_1 = K - \mathrm{reg}_n K, \qquad K_2 = \mathrm{reg}_n K; \tag{1}$$
$$A_1 = \underset{a \in A}{E}[(a) \times B \subset K_1], \quad A_2 = A - A_1.$$

A and B as divisors of a polytope are compact absolute neighbourhood retracts (see [4] § 48, III; we shall use the abbreviation ANR). Since A_2 is obviously open, A_1 is compact.

Let us first prove that

$$\dim B = n - 2. \tag{2}$$

By a known inequality we have $\dim K \leqslant \dim A + \dim B$. Since $\dim A = 2$, then $\dim K \geqslant 1 + \dim B$ [5]. Consequently, $n - 1 \geqslant \dim B \geqslant n - 2$. Suppose $\dim B = n - 1$. Then A_1 is a finite set. For suppose A_1 were to contain

an infinite sequence of different points $\{p_k\}$, $k=1, 2, \dots$ and let $p_0 = \lim_{l=\infty} p_k$. By the compactness of $A_1, p_0 \in A_1$. Then $(p_k)\times B \subset K_1$, $\dim[(p_k)\times B] = n-1$, $k=0, 1, 2, \dots$ and

$$\mathrm{Lim}\,[(p_k)\times B] = (p_0)\times B.$$

It follows that K_1 would contain an $(n-1)$-dimensional subset non-dense in K_1. Since K_1 is an at most $(n-1)$-dimensional polytope, this is impossible. Now denote

$$o(A) = \mathop{E}_{p\in A}[\dim_p A = 2].$$

By the compactness of A we have $\dim o(A) = 2$ (see [4] § 40, V) and it follows that there exists a point $p \in o(A) - A_1$, i. e. $p \in o(A)\cdot A_2$. From theorem 13 in [3] we conclude that $p \in \mathrm{reg}_2 A$. Consequently A contains a closed 2-cell Q_2. But this implies $\dim A\times B \geqslant \dim Q_2\times B = n+1$, and we have arrived at a contradiction. Thus (2) is proved.

Now, we shall prove that

$$A_2 \subset \mathrm{reg}_2 A. \tag{3}$$

By th. 13 in [3] it suffices to show that A_2 is homogeneously 2-dimensional. Suppose, on the contrary, that $a \in A_2$ and $\dim_a A \leqslant 1$. By (1) there exists a neighbourhood U of a in A such that for some set $V \subset B$, $U\times V$ is homeomorphic to an n-dimensional Euclidean region. On the other hand, U can be disconnected by a set of dimension $\leqslant 0$. From (2) it follows that $U\times V$ can be disconnected by a set of dimension $\leqslant n$-2 which is impossible.

We shall now investigate the set A_1. First we prove that

$$\dim A_1 \leqslant 1. \tag{4}$$

By [4] § 46, IV, 3 and the known theorem about Cantor-manifolds it suffices to show that A_1 does not contain a continuum non-dense in A_1. But if C were such a continuum, then $C\times B$ would be a continuum non-dense in K_1 and of dimension $\geqslant n-1$. This, however, is impossible because K_1 is a polytope of dimension $\leqslant n-1$.

Let us now prove the following proposition (due to K. Borsuk):

(5) *There exists an open and connected subset V of B for which*

$$A_2\times V \subset K_2.$$

Let $(a, b) \in K_2$. Then $a \in A_2$ and (3) insures the existence of a connected neighbourhood U of a in A homeomorphic to an open 2-cell. Let V be a connected neighbourhood of b in B for which $U\times V \subset K_2$. We assert that V satisfies (5). In fact let $a' \in A_2$ and U' be a neighbourhood of a' in A homeomorphic to U. Obviously $U'\times V$ contains $(a')\times V$ and is homeomorphic to $U\times V$. Hence $U'\times V \subset K_2$ and (5) is proved. Henceforth by V we shall denote the fixed set satisfying (5). We now prove:

(6) *Given $a \in A_1$, there exists a neighbourhood U_1 of a in A_1 such that for each $b \in V$, $U_1\times V$ is a neighbourhood of (a, b) in K_1.*

For let U be a neighbourhood of a in A. Denote $U_1 = U \cdot A_1$ and let $b \in V$. Then by (1) $U_1 \times V \subset K_1$ and by (5) $(U - U_1) \times V \subset K_2$. It follows that $U_1 \times V = (U \times V) \cdot K_1$; therefore $U_1 \times V$ is an open subset of K_1 containing (a, b).

It follows at once from (6) that A_1 is an ANR. If it were $\dim A_1 \leqslant 0$, then by th. 1 in [6] our theorem would be established. Hence we may assume $\dim A_1 = 1$. We shall prove that

(7) A_1 *is a graph (i. e. a polytope of dimension* $\leqslant 1$).

Because of the local connectedness of A_1 it is enough to show that this is so for each component of A_1 different from a point. Let C be such a component. Denote $C_1 = \underset{a \in C}{E}[(a) \times V \subset K_1 - \mathrm{reg}_{n-1} K_1]$, $C_2 = C - C_1$. Let $p \in C_2$. Then by (6) there exists a neighbourhood of p in C which is the topological divisor of an open subset of $\mathrm{reg}_{n-1} K_1$. Applying th. 3 from [3] we conclude that $C_2 \subset \mathrm{reg}_1 C$. Since a compact ANR containing only a finite number of points of order $\neq 2$ is a graph, (7) will be established if we show that C_1 is finite. And it is, for otherwise it is not hard to verify that $K_1 - \mathrm{reg}_{n-1} K_1$ would contain a non-dense subset of dimension $n-2$.

We shall prove now:

(8) *All save possibly a finite number of points p of A_1 satisfy the following proposition: given $\varepsilon > 0$, there exists $\delta > 0$, such that, if $x \in A_1$, $y \in A_2$, $\varrho(p, x) + \varrho(p, y) < \delta$, there exists in $A_2 + (x)$ an arc $L = xy$ of diameter less than ε.*

Since the components of A_1 are finite in number and in each of them the set C_1 is finite, it is enough to show that (8) holds for every point $p \in C_2$. Suppose $p \in C_2$ and let ε be a positive number. Let $b \in V$ be such that $(p, b) \in \mathrm{reg}_{n-1} K_1$. Then (p, b) belongs to the interior of an $(n-1)$-dimensional simplex of K_1. Denote this simplex by Δ_p. Choose $\eta > 0$ such that if $x \in C$ and $\varrho(p, x) < \eta$, then $(x, b) \in \mathrm{Int}(\Delta_p)$.

By (5), if $y \in A_2$, then $(y, b) \in K_2$. Denote by Δ_y this n-dimensional simplex of K which contains (y, b) in its interior. Choose $\xi > 0$ such, that if $y \in A_2$ and $\varrho(p, y) < \xi$, then Δ_p lies on the boundary of Δ_y. Consequently, if $\varrho(x, p) + \varrho(y, p) < \min(\eta, \xi)$, there exists an arc $M = (x, b)(y, b) \subset \mathrm{Int}(\Delta_y) + (x, b)$. Since $\varrho((x, b), (y, b)) = \varrho(x, y) \leqslant \varrho(x, p) + \varrho(y, p)$, one can admit that the diameter of M tends to zero with $\varrho(x, p) + \varrho(y, p)$. Let $\delta < \min(\eta, \xi)$ be such that if $\varrho(x, p) + \varrho(y, p) < \delta$, then the diameter of M is $< \varepsilon$. Putting

$$L' = \underset{a \in A}{E}\Big[\sum_{b \in B}\{(a, b) \in M\}\Big],$$

we conclude that L' is a Peano continuum with diameter less than ε, contained in $A_2 + (x)$ and containing x, y. Hence L' contains the required arc L and (8) is proved.

We see that (3), (7) and (8) are exactly the same propositions as 1^0, 2^0 and $3^{0\prime}$ of theorem 2 in [6]. Hence by the same theorem A is a polytope and the proof is complete.

In a similar manner one can prove that a 2-dimensional topological divisor of an open subset of a polytope is locally a polytope. Thus, by theorem 3 in [6], we have:

Theorem 2. *Let M be a fibre space with basis B and fibre X. If M is a polytope and if B (or X) is of dimension $\leqslant 2$, then B (or X) is a polytope. (The definition of fibre spaces adopted here is that of* [7] 1.1).

We now prove the following two lemmas (notation as in the proof of th. 1):

Lemma 1. *Let K be n-dimensional polytope and suppose $K = A \times B$ with* $\dim A \leqslant 2$. *Then* $\dim(K - \mathrm{reg}_n K) < n - k$ *implies* $\dim(K - \mathrm{reg}_2 K) < 2 - k$.

Proof. By (1) and (4), $\dim A_1 \times B = \dim A_1 + \dim B \leqslant \dim(A - \mathrm{reg}_n A)$. Hence, by (2), $\dim A_1 < n - k - n + 2$. Since by (3), $A_1 \supset A - \mathrm{reg}_2 A$, the lemma is proved.

Lemma 2. *Let K, A, B be as in lemma* 1. *If* $\mathrm{reg}_n K$ *is connected, so also is* $\mathrm{reg}_2 A$.

Proof. It is easy to verify that the hypothesis implies the connectedness of A_2. The lemma itself then follows from the inclusion $\mathrm{reg}_2 A \subset A_2$.

Because of the corollary to th. 1 in [6] we have, by lemmas 1 and 2:

Theorem 3. *A 2-dimensional topological divisor of a closed pseudomanifold is a closed pseudomanifold.*

We can also prove that a 2-dimensional divisor of a manifold (pseudomanifold) with boundary is itself a manifold (pseudomanifold) with boundary or closed. An analogous theorem also holds for fibre spaces.

Let us finally remark that, by lemma 1, if a connected polytope M satisfying $\dim(M - \mathrm{reg}_n M) < n - 2$ is a fibre space with fibre (base) of dimension $\leqslant 2$, then the fibre (base) is a closed manifold.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. II, No. 7, 1954

EXPERIMENTAL PHYSICS

On the Influence of Low Energy (2-7 MeV) Photons on the Absorption Curve of Extensive Air Showers in Lead

by

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Communicated by H. NIEWODNICZAŃSKI at the meeting of April 26, 1954

Introduction

Measurements of the absorption of extensive air shower particles (which are not μ-mesons or energetic protons) in lead of thicknesses of 10—15 cm., usually performed with single trays of G. M.-counters, have shown their comparatively high penetrating power. Zatsiepin [1], following up the cascade theory of Belenky [2], who took into account the variation of the absorption coefficient of photons with their energy, calculated new cascade curves for lead and obtained for electrons and photons of cosmic radiation a much higher penetration than that which had been assumed previously. Greisen [3] considered the same problem quite independently of Zatsepin, but he, following Hirschfelder et al. [4], [5], assumed also that post-Compton photons lead to a lessening of the effective absorption coefficient of the penetrating low energy (1.2—7 MeV) photons.

Owing to this fact the penetrating low energy photons can be detected under lead absorbers of quite considerable thickness, though the efficiency in detecting them of the usual G. M.-counters is small. This effect was considered by Greisen in an approximate quantitative calculation, in which he evaluated the probability of detecting an extensive air shower particle under an absorber of lead far beyond what is normally considered as the range of the shower. Greisen states that in the last part of the range of a shower in lead the soft component of cosmic radiation is represented mostly by the penetrating low-energy photons. These photons hitting the detector are registered by it mainly through the Compton-effect in its walls. In this way we can detect an extensive shower particle under an absorber of a thickness much greater than that at which the shower has ceased to multiply.

This point of view has been confirmed by Daudin [6] and also recently by Bassi et al. [7]. From their experiments the existence of these photons can be deduced from the absorption of the shower particles in thick layers of lead.

In the experiment described in this paper the author took measurements with the aim not only of finding the penetrating photons produced in lead absorbers by the soft component of cosmic radiation at low altitudes, but also of investigating their influence on its absorption curve in lead.

Apparatus

The apparatus consisted of three G. M.-counter trays A, B, C, each with an area $S = 0.374$ m.², placed in the corners of an equilateral triangle (3.4 m.), and set in coincidence with a telescope P composed of two trays D and E, each with an area $s = 0.291$ m.². The telescope was enclosed by a lead wall 10 cm. thick and a lead base 5 cm. deep, and was covered with an absorber, 3 mm. to 25 cm. thick. The coincidence circuit with a resolving time of c. 3 μ secs. registered simultaneously threefold coincidences $T = (ABC)$, fourfold coincidences $D = (ABCD)$ and $E = (ABCE)$ and fivefold coincidences $P = (ABCDE)$. The roof of a wooden hut was equivalent to 1 g./cm.². The measurements were made in Cracow at an altitude of 229 m.

If we let x denote the density of the electrons of the extensive air shower hitting the absorber, then for the probability of operating the telescope under some absorber by the local shower in it, initiated by extensive air shower particles we can write:

$$1 - e^{-(\varepsilon + \lambda p + km) \cdot x \cdot S \cdot r}, \tag{1}$$

where ε is the probability that an electron hitting the absorber will go through, or give at least one secondary electron which will operate the detector, p is the ratio of photons to electrons in extensive air showers, λ is the probability that a photon hitting the absorber will produce below the latter at least one secondary electron, which will operate the detector, m is the ratio of penetrating particles to electrons in the extensive air showers, k is the probability (~ 1.0) of detection of the penetrating particle falling on the absorber, and rS is the effective surface of the telescope as compared with the surface S of the trays A, B, C.

If we put $(\varepsilon + \lambda p + km) \cdot r = n$, then we can evaluate the rate of fourfold coincidences from the formula

$$C_4 = k\Gamma(-\gamma) \cdot S^{\gamma} \cdot [\Sigma_3(\gamma) + h(n, \gamma)], \tag{2}$$

where γ is the exponent of the integral density spectrum of extensive air shower particles,

$$\Sigma_3(\gamma) = -3 + 3.2^{\gamma} - 3^{\gamma},$$

and

$$h(n, \gamma) = -n^{\gamma} + 3(1+n)^{\gamma} - 3(2+n)^{\gamma} + (3+2)^{\gamma}.$$

Evaluating the ratio between fourfold and threefold coincidences, we can write the expressions

$$\frac{C_4}{C_3} = \frac{\Sigma_3(\gamma) + h(n . \gamma)}{\Sigma_3(\gamma)}. \tag{3}$$

By means of expression (3) we calculate n, from the measured rates of coincidences, n being the relative number of particles beneath the absorber, as compared with the number of electrons falling on the trays A, B or C. For the exponent γ we take the value 1.41 ± 2.02, which we got from the measurements of the ratio between fourfold and threefold coincidences without the absorber.

Results of measurements

Table I gives the relative numbers of the particles obtained from the measurements. In column n_E of Table I we give the relative numbers of the particles registered in the lower tray of the telescope P, in column n_P — the relative numbers of the particles registered in the telescope P and corrected for inefficiency caused by the spaces between the effective surfaces of the counters.

TABLE I

Thickness of absorb. mm. Pb	Relative numbers of particles			$\bar{n}_E - n_P$	$\frac{n_E - n_P}{n_P}$ %
	n_E	n_P	n_E		
0	0.730	0.561	0.594	0.033 ± 0.002	30.1 ± 1.0
3	0.743	0.605	0.604		22.8 ± 0.8
5	0.758	0.617	0.618		22.8 ± 0.9
10	0.649	0.530	0.529		22.4 ± 0.4
17	0.562	0.452	0.458	0.006 ± 0.002	24.4 ± 0.9
25	0.412	0.325	0.336	0.011 ± 0.003	26.8 ± 1.6
50	0.190	0.142	0.155	0.013 ± 0.003	33.8 ± 3.1
75	0.0890	0.0568	0.0725	0.0157 ± 0.0015	56.7 ± 4.0
100	0.0515	0.0346	0.0420	0.0074 ± 0.0008	48.9 ± 2.7
150	0.0282	0.0216	0.0230	0.0014 ± 0.0005	30.5 ± 3.0
200	0.0238	0.0195	0.0194		22.1 ± 3.2
250	0.0235	0.0191	0.0191		23.0 ± 2.6

If we compare the columns n_E and n_P from Table I we perceive some difference between n evaluated from E coincidences (n_E) and n evaluated from P coincidences (n_P). As the threshold energy for registering the electrons in both arrangements is the same, this difference may be ascribed mainly to the difference in solid angles (the difference in r in the expression (1)) of both arrangements, but with greater thicknesses it is due also to the detection in tray E of the penetrating low energy photons, which do not operate the telescope P. The difference at the zero absorber is due also to the small contribution (5.5%) of shower photons not having very

high energies, converted in the walls of the telescope between the trays D and E, and registered in tray E.

By means of the data from column n_E we can evaluate the relative numbers of the particles which would be registered in tray E, if this had registered the extensive air shower particles in the same solid angle as the telescope P. In this way we get a new transition curve $\overline{n}_E$ (Table I), which should be the same as n_P, if the composition of the particles registered in both arrangements (P and E) were the same. We find that both curves are similar for absorbers between 3 mm. and 17 mm. and over 20 cm. Pb. The difference of the two curves (n_P and $\overline{n}_E$) between 17 mm. and 20 cm. is due

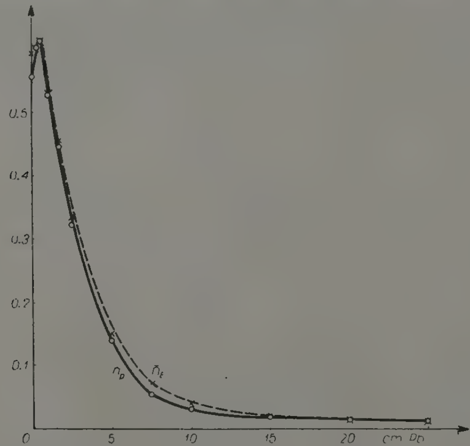

Fig. 1

to the detection of penetrating low energy photons in tray E. Because of the rather high threshold energy of the telescope P for detecting electrons (~ 13 MeV) and the low efficiency (~ 0.02) in the operation of the single tray of counters by the penetrating low energy photons, they can be registered only in tray E.

If we compare the data of the columns n_P and $\overline{n}_E$ in Table I of Fig. 1, we notice a quite distinct difference between the two transition curves. Because of the high penetration of the low energy photons the transition curve $\overline{n}_E$, measured by means of a single tray, has a lower slope than the curve n_P obtained by means of a telescope P, which detects only high energy photons and electrons through their secondary electrons. From the noticeable slope of the transition curve obtained by means of the telescope P beyond 15 cm. Pb we conclude that at low altitudes it is possible to observe in extensive air showers particles, which, if they are electrons or photons, ought to have energies $\gtrsim 10^{12}$ eV.

The difference between the two curves giving the contribution of the registered electrons in tray E due to the low energy photons is shown in column $(n_E - n_P)$ of Table I. It shows a maximum at about 70 mm. Pb. This contribution to the registering of the cascade particles is less than that assumed by Greisen for high altitude measurements

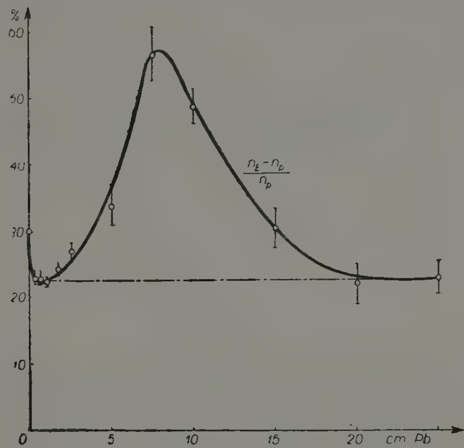

Fig. 2

The contribution of penetrating low energy photons is especially distinct if we evaluate the ratio of $(n_E - n_P)$ to n_P, which may be considered as a measure of inefficiency, as compared with tray E, of the telescope P as regards the registering of extensive air shower particles. The evaluated figures are given in column $(n_E - n_P)/n_P$ and in Fig. 2.

The inefficiency of the telescope as regards registering with both arrangements is due to the difference between the solid angles, and amounts to 22.7%. Comparing the figures in column $(n_E - n_P)/n_P$ we see that the inefficiency of the telescope at the absorbers between 3 mm. and 17 mm. and over 150 mm., may be explained by the difference between the solid angles in both these ways of detecting the extensive air shower particles. The inefficiency of the telescope for thickness of the absorbers between 17 and 150 mm. cannot be attributed to the same cause. It is much higher and shows a maximum at an absorber of about 80 mm. Pb. This high inefficiency must be caused by the penetrating low energy photons which are produced in lead by the absorption of cascade shower particles.

Conclusions

1. The contribution of the penetrating low energy photons to the registering of cascade shower particles at low altitude is observed for lead absorbers above 17 mm. and below 20 cm.

2. The maximum effect of low energy photons in the measurements with a single tray of counters in coincidence with an extensive air showers detector occurs beneath an absorber of about 80 mm.

3. The penetrating low energy photons are not the only cause of the tail of the absorption curve of the shower particles for absorbers above 15 cm. Pb. Measurements with a telescope show that at low altitudes we register a relatively large number of ionizing particles beneath this absorber. If these are the remains of local showers initiated by electrons or photons of cosmic radiation then they ought to have energies $\gtrsim 10^{12}$ eV.

I should like to take this opportunity of expressing my appreciation to Prof. M. Mięsowicz for suggesting this problem, for his active interest in the experiment and for many helpful discussions.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. II, No. 7, 1954

EXPERIMENTAL PHYSICS

The Transition Curve of the Photon Component of Extensive Air Showers

by

J. M. MASSALSKI

Communicated by H. NIEWODNICZAŃSKI at the meeting of April 26, 1954

Introduction

The theories of cascade showers given e. g. by Belenky [1] or by Jánossy [2], tabulated by Jánossy and Messel [3] estimate some excess of the total number of photons over the number of electrons in the electron-photon cascade. In the literature of this subject there are three experimental papers dealing with the problem of the ratio of the number of photons to the number of electrons in extensive air showers. Millar [4] obtained the value 0.25 for this ratio. Bassi et al. [5] have about the same value. But Milone [6], whose results were published just when the experiment described in this paper had been carried out, gave for the same ratio a value of about 1.0. It seems that all these measurements, except a series of measurements by Milone, were made in almost identical conditions rather near the shower cores.

The results obtained by Bassi et al. and by Milone concern particles with a threshold energy of the order of 10 MeV., but those of Millar concern particles with a threshold energy of 140 MeV.

In view of these facts it seemed to us important to take measurements of the transition curve of the photon component of extensive air showers with an apparatus registering the shower particles with large effect of both electrons and photons.

Apparatus

The apparatus consisted of three trays of G. M.-counters A, B, C, each with an area $S = 0.374$ m.^{2}. Their centres were put in the corners of an equilateral triangle with a side of 3.4 m. In the centre of this triangle was placed a telescope consisting of two trays D and E, lying one above the other, each with an area $s = 0.291$ m.^{2}. The trays A, B nd C, each contained 8 all metal counters, 90×5.2 cm.^{2} in dimension and

made of brass tubes 1 mm. thick. The trays D and E each contained 8 counters, 70×5.2 cm.^{2} in dimension, and of the same wall thickness. The telescope, which was placed in a wooden hut, was surrounded by a lead shield 10 cm. thick at the sides and ends, and 5 cm. underneath. The wooden covering of the hut was about 1 g./cm.^{2}. The measurements were made (in Cracow at an altitude of 229 m.) with a coincidence set of a resolving time of c. $3\,\mu$ secs. The number of threefold accidental coincidences did not exceed 1%.

The four *P. O.* registers counted four different kinds of coincidences simultaneously: $(ABC)=T$, $(ABCD)=D$, $(ABCE)=E$, $(ABCDE)=P$. The lead adsorbed was put either between the trays D and E (we denote the coincidences in this case by D_m, E_m, P_m) or above the telescope (we get in this way the coincidences D_n, E_n, P_n).

The principle of measurements

If we admit the Poisson distribution of the density of extensive air shower particles and their differential density spectrum of the form $Kx^{-(\gamma+1)}dx$, then we can evaluate the rate of threefold coincidences by means of the formula

$$(1)\quad \begin{aligned} T=(ABC)&=\int_0^\infty(1-e^{-Sx})^3 Kx^{-\gamma-1}dx = K\Gamma(-\gamma)S^\gamma(-3+3\cdot 2^\gamma-3^\gamma)= \\ &=AS^\gamma\Sigma_3, \quad \text{where} \quad \Sigma_3=-3+3\cdot 2^\gamma-3^\gamma. \end{aligned}$$

The ratio D/T for the zero absorber depends only on the exponent γ. From all the series of our measurements of D/T we obtain for γ the value 1.41, which is in close accordance with the values given by other authors.

Let us consider the case when a lead absorber is put between the trays D and E.

If the average density of the ionizing shower particles (electrons) falling on the absorber is x, then we can express the probability of operating tray E as: $1-e^{-(\varepsilon+\lambda p)rSx}$, where ε is the probability that an ionizing shower particle falling on the absorber will produce at least one secondary ionizing particle passing through tray E; λ is the probability that a shower photon falling on the absorber will produce at least one secondary electron going through tray E; p is the ratio of the number of photons to the number of electrons in the showers falling on the apparatus, and rS is the effective area of tray E. We can evaluate the rate of the coincidences E_m from the formula:

$$(2)\quad E_m=\int_0^\infty(1-e^{-Sx})^3(1-e^{-(\varepsilon+\lambda p)xrS})\,Kx^{-\gamma-1}dx = AS^\gamma\{[\Sigma_3+h(\varepsilon+\lambda p)r]\},$$

where

$$h(t)=t^{-\gamma}+3(1+t)^\gamma-3(2+t)^\gamma+(3+t)^\gamma.$$

From formulas (1) and (2) we get

$$\frac{E_m}{T} = \frac{\Sigma_3 + h[(\varepsilon + \lambda p)r]}{\Sigma_3}. \tag{3}$$

Fig. 1 shows the values of E_m/T as a function of $(\varepsilon + \lambda p)r$.

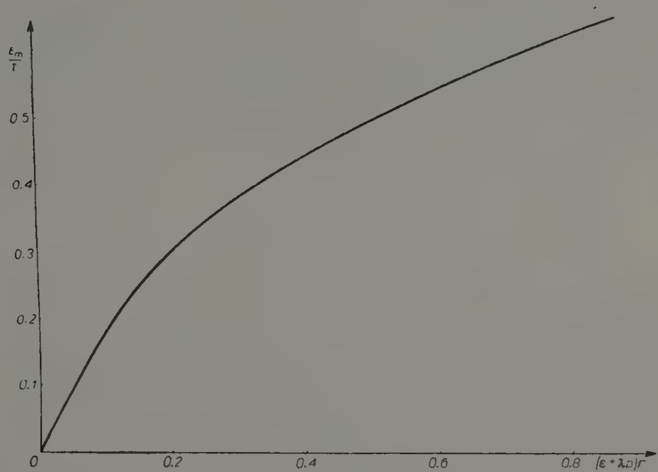

Fig. 1

The rate of coincidences E given by the single shower photons, i. e. those obtained when out of the two trays D and E only tray E is operated, may be considered as the rate of the anticoincidences $E_a = E_m - P_m$.

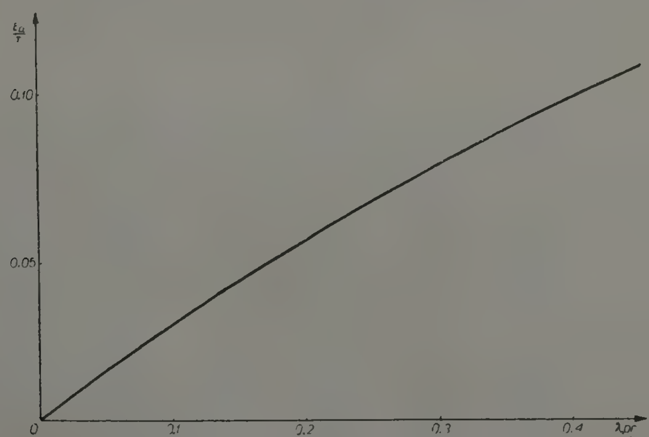

Fig. 2

$$E_a = \int_0^\infty (1-e^{-Sx})^3 (1-e^{-\lambda prSx}) e^{-rSx}) K x^{-\gamma-1} dx = AS^\gamma \{h[r(1+\lambda p)] - h(r)\}$$

$$E_a = \frac{\{h[r(1+\lambda p)] - h(r)\}}{\Sigma_3}. \tag{4}$$

By means of the last formula we can directly evaluate λpr. Fig. 2 shows the dependence of E_a/T on λpr as given by (4). From the data of the measurements of E_m/T and the relation shown in Fig. 1 we can evaluate $(\varepsilon+\lambda p)r$, and from the difference $E_m/T - P_m/T = E_a/T$ we can evaluate λpr. Thus, considering the figures E_m/T and P_m/T, we can obtain values of εr and λpr as a function of the absorber thickness.

The results of measurements

Table I shows the results of measurements of P_m/T and E_m/T for different thicknesses of lead absorbers put between trays D and E, and of P_n/T for the same absorbers put over the telescope. The counted number of coincidences T in all the series of measurements for every absorber thickness was about 5000. There is some difference between the values P_m/T and E_m/T (0.119 against 0.128) for an absorber with a thickness of 10 cm. We may suppose that the difference between P_n/T and E_m/T is caused mainly by the side and scattered particles which are registered in tray E but not in the telescope. If we assume that the percentage contribution of these particles is the same for all absorber thicknesses below 10 cm., then we can make the correction for this effect by increasing all the values of P_m/T by 7%. In this way we obtain the values P/T and we can evaluate the difference $E_m/T - P/T = E_a/T$. Putting the data of E_m/T and $E_m/T - P/T$ into formulas (3) and (4) we evaluate $(\varepsilon+\lambda p)r$, λpr and εr. The values obtained in this way are given in Table I.

TABLE I

mm. Pb	P_n/T	P_m/T	E_m/T	P/T	$E_m/T - P/T$	$(\varepsilon+\lambda p)r$	λpr	εr
0	0.523	0.523	0.596	0.568	0.028	0.725	0.095	0.630
5	0.536	0.489	0.580	0.513	0.067	0.679	0.250	0.429
10	0.513	0.455	0.543	0.490	0.059	0.600	0.215	0.385
17	0.469	0.430	0.511	0.456	0.055	0.516	0.200	0.316
25	0.386	0.376	0.443	0.407	0.036	0.391	0.125	0.266
50	0.226	0.238	0.264	0 258	0.006	0.168	0.018	0.150
75	0.114	0.152	0.169	0.162	0.007	0.090	0.023	0.067
100	0.072	0.119	0.128	0.128	0.00	0.063	0.000	—

The last two columns of Table I give the transition curves of shower electrons and photons. From these data we obtain the value 0.5 for λp at the maximum of the transition curve.

Arley [7], assuming a certain spectrum of the energies of the electrons and photons of total cosmic radiation, obtained for $\lambda_{\max}$ the value 0.5. It is doubtful whether the evaluations made by Arley are applicable here. It is especially doubtful whether the spectrum of photons in air showers may be taken as being the same as that assumed by Arley for total cosmic radiation. In spite of these doubts, we use the values given by Arley only

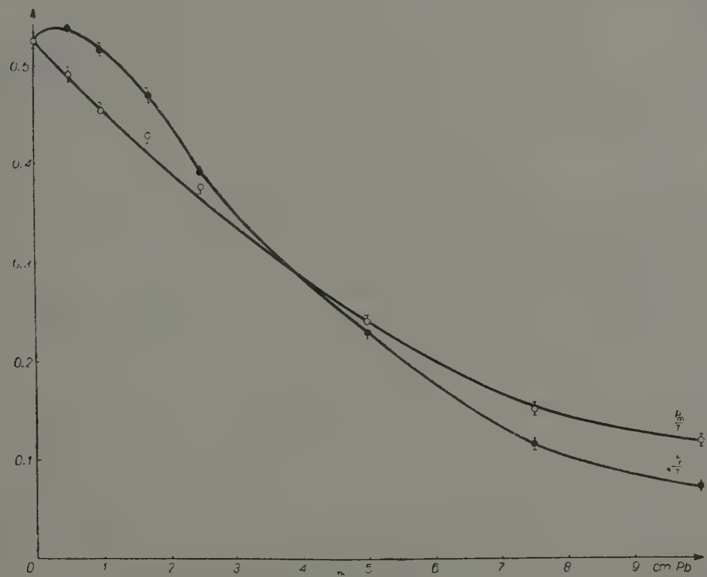

Fig. 3

as a conventional way of comparing the value of p obtained in this experiment with the values obtained by other authors who have followed up his evaluations. Thus, from our measurements we get for p the value 1.0 ± 0.3.

Besides the series of measurements of P_m/T, another series P_n/T was made with an absorber over the telescope. For the discussion of the results obtained from the measurements of P_m/T and P_n/T we may use expressions (3) and (4). We do not need any normalization for the discussion of these results, as both series P_n/T and P_m/T have the same value for the zero absorber. According to formula (4) we ought always to have the relation $P_m/T < P_n/T$. As we can see from Fig. 3, this relation is not fulfilled for all the absorber thicknesses. With an absorber of 4 cm. Pb the curve P_m/T intersects the curve P_n/T. This fact shows that our expressions do not take into account all the effects which contribute to the measure-

ments of P_m/T, and which do not take place in the measurements of P_n/T. To register the coincidence P_n/T, it is necessary that the electron going out of the absorber should have an energy of at least 13 MeV., enough to penetrate three walls of the telescope counters, whereas in the measurements of P_m/T the coincidence is registered if the photon or electron comming out of the absorber causes a discharge in tray E (one counter wall) simultaneously with the discharge initiated by an electron in tray D lying over the absorber.

The difference in the threshold energies of the apparatus for registering the shower electrons in both series of measurements as well as the difference in the mechanism of operating the telescope in both cases [8]—[10] give a higher contribution to the ratio of coincidences P_m/T as compared with the transition effect of the shower photons in the measurements of P_n/T. Thus, in spite of the fact that both series of measurements are made with the same telescope, they cannot give correct values of ϵ and λp. But, on the other hand, a discussion of the results obtained for measurements with an absorber between 0 and 4 cm. Pb might be very valuable, as it undoubtedly gives the lower limit of the value of p. The value of p obtained in this way is 0.8.

It is a pleasure for the author to express his thanks to Prof. M. Mięsowicz for suggesting this problem and for many valuable discussions.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. II, No. 7, 1954

CHEMISTRY

On the Positive-Negative Azeotropes Formed by Naphthalene, Cresols and Pyridine Bases. XIX

by

K. ZIĘBORAK and H. MARKOWSKA-MAJEWSKA

Presented by W. ŚWIĘTOSŁAWSKI on March 29, 1954

Introductory remarks

The purpose of these investigations was to examine by the ebulliometric method [3], [4] three systems composed, on the one hand, of naphthalene and a cresol fraction F, boiling at practically constant temperature 202° C., and on the other, of three fractions of pyridine bases P_1, P_2 and P_3, each used separately.

The boiling temperature ranges of the three fractions were: P_1: 142—145° C.; P_2: 157—157.5° C. and P_3: 163—164° C. Experiments have shown that the isobars obtained by mixing each of the base fractions separately with a cresol fraction F, are each characterized by a maximum boiling temperature. The maxima corresponded to mixtures behaving as if each of them consisted of only a binary negative azeotrope.

Experimental part

In Table I are listed: the composition, the boiling temperatures and the increases in azeotropic boiling temperature.

TABLE I

Maximum boiling temperature of mixtures $[(-)F, P_1]$, $[(-)F, P_2]$, $[(-)F, P_3]$

Mixture	Weight percent of P at maximum	Maximum boiling temp. $t_{(-)F,P}$	Azeotropic temp. increase $t_{(-)F,P} - t_F$
$F+P_1$	10%	202.5	+0.5
$F+P_2$	20%	204.4	+2.4
$F+P_3$	22%	204.9	+2.5

In Fig. 1 points C_1, C_2 and C_3 correspond to the compositions of mixtures $[(-)F, P_1]$, $[(-)F, P_2]$ and $[(-)F, P_3]$, each being characterized by a maximum boiling temperature.

Table II contains the figures obtained from an ebulliometric examination of the boiling temperature isobars formed by naphthalene (H) together with the mixtures C_1, C_2 and C_3. In columns II, III and IV of Table II are listed: the weight percentages of naphthalene at the minimum boiling temperatures M_1, M_2 and M_3 and the decreases in boiling temperatures, when compared with $t_{(-)F,P}$, found for mixtures represented in Fig. 1 by points C_1, C_2 and C_3.

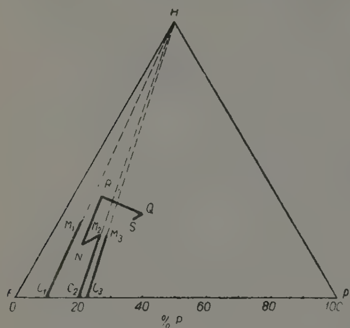


Fig. 1. Projection on the concentration triangle of the maximum boiling points C_1, C_2, C_3 and the minimum ones M_1, M_2, M_3 of mixtures of (F, P_1, H), (F, P_2, H) and (F, P_3, H), where H stands for naphthalene. Points C_2, M, N, R, O and S correspond to the various compositions of mixtures. The thick straight lines are the projections of isobars examined by the ebulliometric method

TABLE II

Minimum boiling temperatures of naphtalene mixed with (F, P_1), (F, P_2) and (F, P_3)

I	II	III	IV
F, P_1+H	10	202.48	— 0.02
F, P_2+H	18	204.03	— 0.37
F, P_3+H	21	202.39	— 0.51

In addition to examination of the shape of the boiling temperature isobars, the corresponding sections [1] HC_1, HC_2 and HC_3 of the main lines are graphically represented in Fig. 1 by thick lines. Other isobars were examined for the purpose of determining the position of the azeotropic points. For instance, lines MN, NR, RQ and QS represent the isobars examined for mixtures composed of H, F and P_2 (Fig. 1). ..

Discussion of the results

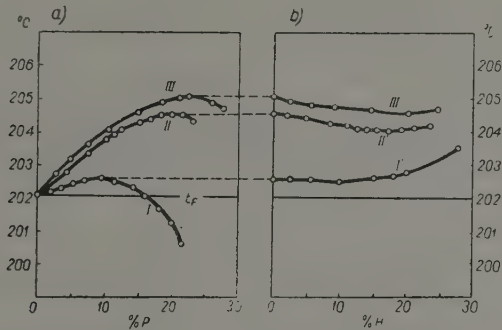
The conclusions drawn from these experiments may be formulated as follows:

1. All of the ternary positive-negative azeotropes [2] formed by naphthalene, meta and para-cresols and the picolines and lutidines found in the three fractions P_1, P_2 and P_3 belong to the almost tangent type.

2. The tridimensional surfaces of the isobars of all these azeotropes, and also of their mixtures are very flat near points C_1, C_2 and C_3.

3. Points t_{Az}, which correspond to the compositions of the ternary saddle azeotropes under consideration, and also all the three top-ridge lines lie inside triangle HCP.

4. The boiling temperatures of all the azeotropes involved differ so little from each other that their separation by an effective distilling column seems to be impossible.



Figs. 2a and 2b. Curves I, II, III represent the boiling temperature isobars of mixtures of a cresol fraction F with each of the three pyridine bases. I′, II′, III′ are the corresponding isobars formed by mixtures of F, P_1, F, P_2, F, P_3 with naphthalene

In order to prove these conclusions, a mixture of naphthalene (17.5%), a fraction of meta and para-cresols (62.5%) and a fraction P_1 (142—145° C.)

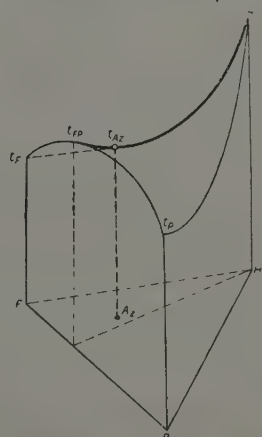


Fig. 3. Tridimensional model of a saddle azeotrope composed of naphthalene (H), one of the cresols (F) (meta or para) and one of the constituents of pyridine bases fraction P_2. Point t_{Az} represents the boiling temperature of this azeotrope. This point is situated on the top-ridge line, which lies in the vicinity of the main line [2]

of pyridine bases (20.0%) was submitted to fractional distillation on a thirty-five plate column. The main fraction, corresponding to 75% of the charge, was collected within 202.5—203.7° C., and the average contents of these mixtures were found for tar bases 10.4%, for naphthalene 20.6% and for

cresols 69.0%. It should be noted, that fraction P_1 used in this experiment was characterized by the largest boiling-range of temperature, 142—145° C., compared with the two others, P_2 and P_3. For comparison, in Figs. 2 a and 2 b, two series of boiling temperature isobars I. II, III and I′, II′, III′, are given. A corresponding explanation is given in the legend of the two drawings.

One should note that curve I′ is almost tangent to the horizontal line drawn through point t_M, and that the two other isobars are slightly concave, showing relatively small depression of the boiling temperatures.

This is certaintly in agreement with the figures listed in the last column of Table II.

A tridimensional model of a ternary saddle azeotrope $[(-)F, P(+)H]$ is shown in Fig. 3. The system is composed of naphthalene, one of the cresols (meta or para-isomer) and one of the components of fraction P_2.

We wish to express our thanks to Prof. Świętosławski for his help and advice.

Summary

1. The ebulliometric method was used for determining the shape of the boiling temperature isobars of systems composed, on the one hand, of naphthalene and a fraction F of m and p-cresols, characterized by the constant boiling temperature, $t_F = 202°$ C., and, on the other, of fractions of pyridine bases characterized by the following temperature ranges: P_1: 142—145° C.; P_2: 157—157.5° C.; P_3: 163—164° C.

2. It was found that all three isobars were characterized by minimum boiling temperatures, indicating that each of the mixtures was composed of a corresponding number of three-component positive-negative (saddle) azeotropes, characterized by boiling temperatures differing very little from one another.

3. The boiling temperatures of the three mixtures of positive-negative azeotropes formed by naphthalene, with a fraction F of m and p-cresols, and of the three pyridine bases P_1, P_2 and P_3 differed slightly from the maximum boiling temperatures formed by fraction F mixed with P_1, P_2 or P_3 respectively.

4. The tridimensional surface corresponding to the boiling temperature isobars was examined by the ebulliometric method, and it was found that in the vicinity of the minimum boiling temperature lying on the top-ridge line [2], these surfaces were very flat. Therefore, it may be said that the behaviour of each of the three above-mentioned polyazeotropic mixtures was similar to that of a single ternary saddle azeotrope composed of three individual substances.

5. The formation of a polyazeotropic mixture consisting of a number of ternary saddle azeotropes $[(-)F, P(+)N]$, F, P and N representing one of the cresols, one of the pyridine bases and naphthalene respectively, takes place in the course of distillation of the carbolic and middle oils of coal tar.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. II, No. 7, 1954

CHEMISTRY

The Applicability of Traube's Rule to the Inhibiting Action of Fatty Acids on Stress Corrosion of Steel

by

M. ŚMIAŁOWSKI and T. OSTROWSKA

Communicated by M. ŚMIAŁOWSKI at the meeting of April 26, 1954

Various authors have asserted the ability of some organic compounds to minimise the corrosion process of metals. Apart from the wide use in practice of inhibitors for reducing the corrosion rate of steel in acids [1], the inhibiting action of some organic compounds in neutral, or almost neutral, media is also known. For instance, great inhibiting effectiveness is shown in these conditions by benzoic acid [2] or by some aliphatic compounds possessing at one end of the molecule a polar group or a group capable of ionisation [3]. The above observations, however, concern the usual forms of corrosion, which attack, more or less uniformly, the whole surface of the metal in contact with the solution. There is considerably less data available on the subject of inhibitors of stress corrosion which causes intercrystalline or intracrystalline cracks in the structure of the metal.

In the course of our experiments on the corrosion cracking of mild steel under the influence of ammonium nitrate solutions, we stated the inhibiting action of fatty acids, an action depending in an interesting way, and in a way which throws light on the mechanism of the phenomenon of inhibition, on the length of the hydrocarbon chain of the fatty acid.

Conditions and results of experiments

Investigations were carried out, using samples of iron wire, 0.45 mm. in diameter, with components in the following percentage quantities:

C	Mn	Si	P	S	Cu	Ni
0.09	0.35	0.00	0.028	0.048	0.47	0.03

Samples, 500 mm. in length, were cut from a coil of cold-drawn wire. These, after being straightened, were heated for half an hour in a hydrogen at-

mosphere at 900° C. In this state of thermal treatment, the tensile strength of the samples equalled 26.0 kG./mm.².

The investigated sample of wire *a* was fastened at one end to a glass hook *b* on the bottom of a narrow vesel *c* (Fig. 1), into which was poured 45 ml. of a 6 N solution of ammonium nitrate. This vessel was provided with a cover *d*, through which water vapour with a temperature of 100° C. was allowed to pass. The other end of the wire was loaded with a weight of 3 kG. The stress calculated per surface unit of a cross-section of the wire sample amounted, therefore, to 18.7 kG./mm.² i. e. about 70% of the tensile strength. The experiments consisted in observing the time required for the wire to break. Fatty acids were introduced into the solution of ammonium nitrate in the form of solutions in methyl alcohol.

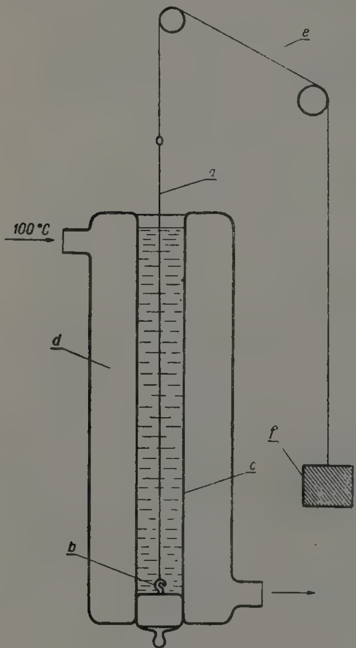

Fig. 1

Fig. 2 shows the relation between the corrosion time t (in minutes) and the concentration c (in moles per litre) for three monocarboxylic fatty acids with an unbranched chain: butyric (3), hexanoic (2), and heptanoic (1).

The course of the curves shows that the greater the number of carbon atoms in a molecule of fatty acid the smaller the critical concentration necessary for reducing the rate of stress corrosion cracking.

Fig. 3 represents the relation between the corrosion time t in the ordinary scale (in Fig. 4 — in the logarithmic scale) and the number of carbon atoms n in a molecule of fatty acid. The acid concentrations in these experiments were higher than the critical ones, i. e. they corresponded to the values, where for a given fatty acid no further increase in inhibition was observed when the inhibitor concentration was increased.

Discussion

The results presented in Figs. 3 and 4 show that the extent to which fatty acids can inhibit the stress corrosion cracking process of mild steel under the influence of ammonium nitrate depends on the length of the hydrocarbon chain, in a way conforming to the rule of Traube [4] and Szyszkowski [5], which was formulated for adsorption on the surface of

aqueous solutions and tested by Freundlich [6] and Claesson [7] for the ability of some solid adsorbents to adsorb fatty acids.

This fact throws light on the mechanism of the process, whereby stress corrosion of mild steel in ammonium nitrate is inhibited by fatty acids. Molecules of fatty acid, becoming adsorbed on the active centres of the steel surface, make them hydrophobic and this, since it renders access of the corroding medium difficult, contributes to a slowing-down of the corrosion process.

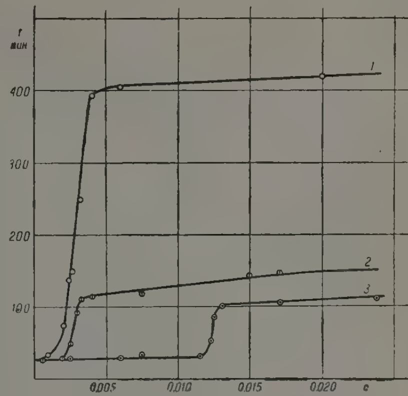

Fig. 2

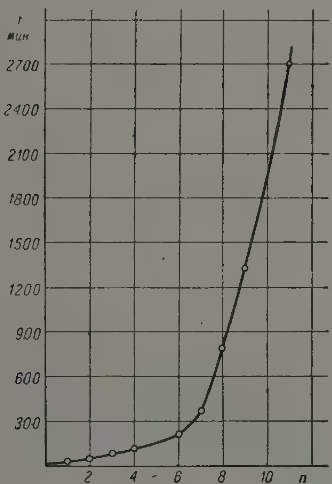

Fig. 3

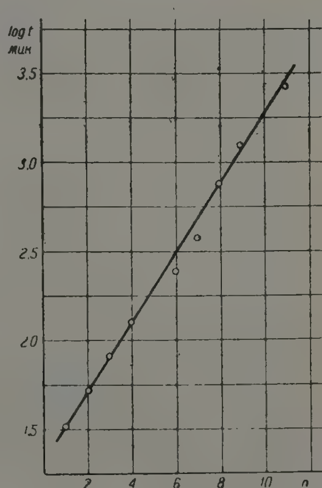

Fig. 4

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. II, No. 7, 1954

CHEMISTRY

The Deformation of Steel under the Influence of Internal Blisters and Cracks Produced by Cathodic Hydrogen

by

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Presented by M. ŚMIAŁOWSKI on May 17, 1954

In the course of further investigations concerning the mechanism of cathodic hydrogen penetration into iron and steel, measurements were made of the deformations of annularly bent (Fig. 1) steel samples under the influence of their saturation by cathodic hydrogen in a solution of 1 N sulphuric acid, with the addition of $2\,10^{-5}$ mole As per litre.

Rings were cut from a cold-rolled steel tube. The chemical composition of the steel was as follows:

C	Mn	Si	P	S
0.26	0.52	0.22	0.015	0.029%

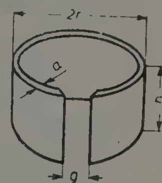

Fig. 1

The dimensions of the samples used for the main series of measurements were as follows: $2r = 38$, $h = 15$, $a = 0.4$, $g = 0.5$ mm. The sample, after having been polished, degreased and covered on its outer surface with a thin layer of ceresine, was connected with the negative pole of a source of direct current and immersed in a small glass vessel, through which flowed the electrolite at a constant temperature of $25 \pm 0.1^{\circ}$ C. The anode, made of platinum sheet, was placed in the middle of the ring. The dimensions of the anode were as follows: $10 \times 10 \times 0.05$ mm. The cathodic current density, calculated on the active, inner surface of the ring, amounted to 0.1 amp./cm.^{2}.

The experiments consisted in removing the ring from the solution every 30 minutes and in measuring under a microscope, always in the same place, the width of the gap g, which usually increased during electrolysis. This phenomenon occured also when the outer surface of the ring was not covered with ceresine. If, on the other hand, the anode was on the outside of a ring not insulated with ceresine, then the width of the gap g during electrolysis most often diminished to zero and the ring contracted with great force.

In the initial state, which was characterised by a considerable degree of cold work, the rings dit not become deformed under the influence of

hydrogenation; the width of the gap g, even after a long period of electrolysis, remained constant or, at the most, increased to 0.04 mm. On the other hand, rings heated at 450° C., or higher, gradually expanded under the influence of hydrogenation; the width of the gap g, after about 2 hours of current flow, acquired a constant value on the average about 2 mm. greater than the initial width. The typical form of the curve representing the changes in width of the gap (Δg in millimetres) in the function of time (t in minutes) is presented in Fig. 2, while Table I contains the values obtained for different samples heated for 2 hours *in vacuo* (at a pressure of 10^{-3} mm. Hg) at different temperatures, and afterwards slowly cooled.

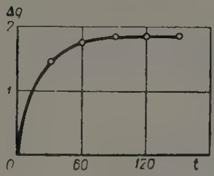

Fig. 2

From Fig. 2 it can be seen that the course of deformation of steel rings, depending on time, is of the same character as that of iron wires and spirals, as was described in preceding papers [1]; namely, after a certain period of electrolysis, it is observed that a maximal degree of deformation is reached, which does not increase further.

TABLE I

Annealing temperature °C	Deformation Δg in mm. after the period of saturation in min.				
	30	60	90	120	150
20	0.020	0.020	0.015	0.030	0.020
	0.015	0.019	0.010	0.010	0.040
	0.010	0.010	0.010	0.060	0.050
450	1.555	2.055	2.395	2.610	2.465
	1.450	1.725	1.780	1.800	1.855
	1.690	1.935	2.010	2.125	2.130
	0.660	1.055	1.260	1.285	1.290
550	1.150	1.835	2.100	2.145	2.150
	1.225	1.742	1.820	1.830	1.835
	0 950	1.542	1.720	1.950	1.975
	1.400	2.030	2.125	2.215	2.220
650	1.020	1.150	1.320	1.420	1.425
	1.220	1.350	1.720	1.920	1.935
	0.920	1.450	1.820	2.120	2.150
900	1.167	1.615	1.725	1.872	1.880
	1.615	1.735	1.872	1.950	1.975
	1.905	2.265	2.435	2.438	2.440
	1.745	1.970	2.245	2.335	2.338

The values given in Table I show that there are great deviations in the values of the observed deformations for a given annealing temperature and therefore it is not possible to estimate the relation between the degree of deformation and the annealing temperature. As there was some conjecture as to the possible passivation of the steel surface under the influence of heating in air under low pressure, a second series of experiments was carried out on rings annealed for 2 hours at 900° C. in different kinds of atmospheres. The rings used for these experiments were of the same quality of steel as before, but of different dimensions: $2r = 24$, $h = 20$ mm. The results given in Table II prove that by annealing in different kinds of atmospheres similar degrees Δg of deformation are obtained; however, in this case, too, great divergences in the results make it impossible to assess the quantitative relations.

TABLE II

Atmosphere and pressure mm. Hg	Deformation Δg in mm. after the period of saturation in min.			
	30	60	120	180
nitrogen	—	1.565	1.705	1.805
760	0.140	0.187	1.050	2.665
hydrogen	1.580	1.780	1.825	1.875
760	1.555	1.410	1.485	1.510
	—	1.400	1.450	1.505
air	1.420	1.530	1.850	1.855
10^{-3}	1.125	1.180	1.220	1.225

With the purpose of studying the mechanism of the observed phenomena and the causes for the divergent results, metallographic examinations of the rings were made both before and after impregnation with hydrogen. Some micrographs are given in Figs. 3—6.

Figures 3 and 4 present micrographs of the transverse section of a ring saturated with cathodic hydrogen after annealing at 450° C. The cracks in the steel in this state of heat treatment were, as can be seen from the micrographs, narrow and long. They ran mostly across crystals and were distributed very irregularly over the whole section of the ring.

In Figs. 5 and 6 are shown micrographs of rings saturated with hydrogen after having been annealed at 900° C. Fig. 5 shows the part of the inner surface of a ring, from which the cathodic hydrogen had commenced its action. Intercrystalline cracks are visible, whereas the internal fissures in Figs 5 and 6 run mostly through the grains.

Discussion

The results of the experiments on the deformation of steel samples under the influence of cathodic hydrogen confirm the view expressed in preceding papers [1], namely, that this phenomenon is due to the accumulation, under high pressure, of molecular hydrogen in the internal structure of the steel.

Owing to the fact that, in contradistinction to palladium*), iron does not show any important changes in electric conductivity under the influence of hydrogenation, and also that the changes in its magnetic properties are, for the most part, insignificant one can assume that the form in which hydrogen diffuses through the iron network is not a proton but an atomic form.

When the hydrogen atoms, in the course of their migration (under the influence of the pressure gradient) through the crystals of iron, encounter e. g. a nonmetallic inclusion or a hole present on the boundaries of the grains, they unite into particles which remain motionless in a given place.

Approximate calculations, based on the work done up to the present, seem to show that the magnitude of the pressures created in the steel structure by the hydrogen blisters arising in this way may probably attain 10^4 or 10^5 atmospheres.

Further experiments aimed at an accurate estimation of this value are at present being prepared.

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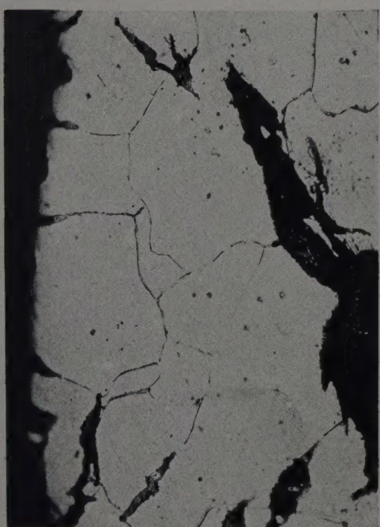
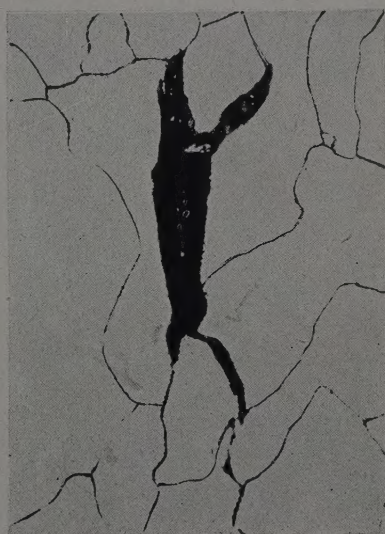
*) It is worth noting that Lewis, Roberts and Ubbelohde [2] come to the conclusion that, under the influence of impregnation with cathodic hydrogen, palladium assumes a state somewhat similar to that obtained by cold work.

Cracks caused by hydrogen in steel annealed at 450° C.

Fig. 3. Non-etched (× 100)

Fig. 4. Etched (× 380)





Cracks caused by hydrogen in steel annealed at 900° C.

Fig. 5. Etched (× 380)

Fig. 6. Etched (× 380)